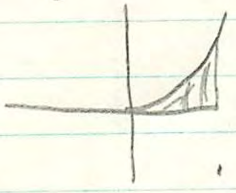
EXAM CH 9 ●●●●●●●●●●● # 1 R/C Pg 362 - 366 ●●●●●●●●●●● # 2 Pg 366 1 - 3 3/17	# 3 Pg 366 4, 7, 8–13,16 also Draw cross sections # 4 R/C Pg 367 - 371 # 5 Pg 371 -372 1 - 5 **the Water Glass problem** ***See lecture notes down*** # 6 Pg 372 9, 14 – 18 **Have pg 366 9 - 13 put up on boards** **Use calculators to get numeric answers.** 3/18	3/19	# 7 Pg 373 1 - 5, 8, 11, 12, 19 ***Also compute using grapher*** **Definite Integrals allow students to finish HW #6** 3/20	# 8 Pg 374 21, 31 - 35 ●●●●●●●●●●● # 9 R/C Pg 375 - 379 **Washers** 3/21
# 10 Pg 379 1 - 8, 9 - 12, 19 ***Also compute using grapher*** ***Important Lecture*** **Shells** 3/24	# 11 Pg 383 1 - 6, 9 Integrate answers to 9. Reconcicle differences ●●●●●●●●●●● # 12 "Voting Power" (an application of integrals) 3/25	# 13 Pg 391 1 - 17, 22, 27 ●●●●●●●●●●● # 14 R/C Pg 397 - 401 3/26	# 15 Pg 410 1 - 7 **Important Lecture length of curve** 3/27	# 16 Pg 410 9, 10, 12, 13 ●●●●●●●●●●● # 17 R/C pg 411-414 **Speed on Curve** 3/28
# 18 Pg 414 1–6,8,10, 14, 15 **do pblm 15 *using a single* integral** $A = \int r^2 \; d\theta$ **area with polar equations** 3/31	# 19 Pg 420-421 1 - 8 **Important Lecture** **Surface of Revolution** 4 / 1	**Important Lecture** # 20 Pg 427 - 428 1, 2, 3, 19, 20, 25, 26 **Simpson's Rule** 4 / 2	# 21 ReWrite ENTIRE "Summary" pg 429-432 # 22 Pg 421 22, 24, 25 GQ Pg 432 1, 3, 9 - 11 4 / 3	# 23, 24, 25 **"Obvious ...."** 4 / 4
# 26 Pg 432-436 1 - 4, 6 - 8, 25, 28, 30, 32, 4/7	# 27 Pg 432-436 39-41, 42a, 43 - 48 Discuss "Download" 4/8	# 28 Pg 432-436 50, 51, 53, 54, 61 4/9	Put Last Year's Exam on the ChalkBoards 4/10	**Calculus Chapter 10** Exam 4/11
Begin AP prep work!!!! Easter Vacation 4/15	Day after exam Ch9 lecture: Hw#2, #2. Show pg 366 1,2,3 then do 9,10. "this y value minus that y-value" if cross sect changes in its nature ..." 4/16	4/17	4/18	**2003**

10.1
Pg 362-367

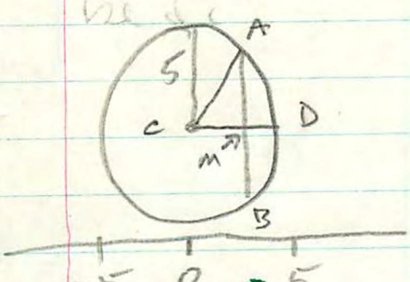
Setting up an integral (easy cross sectional length)

when we are trying to find the area of a two dimensional object we use an integral of a typical cross sections. eg $\int_0^1 x^2 dx$ for



there is nothing sacred about the x or y axis, all we really need is an easy cross section measurement. Then we sum all such cross sections.

example #1 pg 363



we want to set up the integral which will describe the area of this circle.

1st Define C(x) to be the cross sectional length AB*. Draw CA, Draw and CD // to x axis, CD intersects AB at m.

*
C(-5) = ? 0
C(0) = ? 10
C(5) = ? 0

So $\int_{-5}^{5} c(x)\,dx =$ area required.

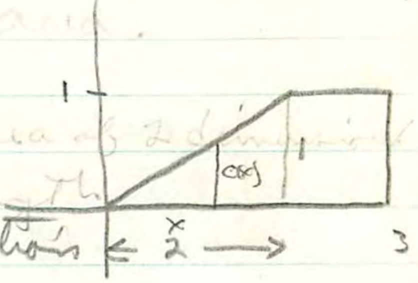
AC = 5 (a radius) $\overline{CM} = |x|$ so $AM = \sqrt{25-x^2}$

Since $AM = \frac{1}{2} AB$ the cross section length is $2\sqrt{25-x^2}$ so area of circle is

$$2\int_{-5}^{5} \sqrt{25-x^2}\,dx$$

10.1
Pg 362-367

example 2

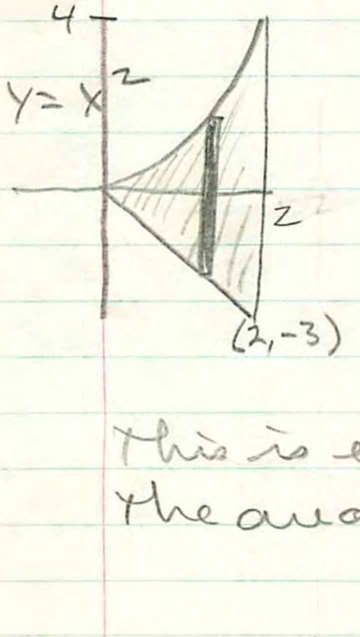


Set up definite integral for:

interval [0, 3]. Notice the cross section changes "in its nature" at x = 2.
Let c(x) be the typical cross section ie a vertical line. $\frac{c(x)}{x} = \frac{1}{2}$ $\therefore c(x) = \frac{1}{2}x$
for the interval [0, 2] $c(x) = 1$ for [2, 3]

$$\int_0^2 \frac{1}{2}x\,dx + \int_2^3 dx$$

example 3



In this example the cross section is the diff between the two curves, ie $Y = x^2$ and $Y = \frac{-3}{2}x$ (found using the two pts (0, 0) and (2, -3)). The distance abv the x axis is x^2. The distance below the x axis is $\frac{3x}{2}$. We need to add these to make up the entire cross section length

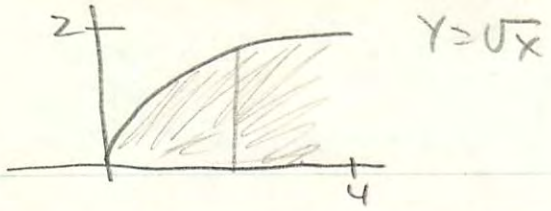
This is equivalent to $(\overset{\text{to get area above x}}{x^2}) - (\overset{\text{to get area below axis.}}{-\frac{3}{2}x})$ so the area will be $$\int_0^2 x^2 + \frac{3x}{2}\,dx$$

Generally we can consider if $f_1(x)$ and $f_2(x)$ are borders ($f_1(x)$ above $f_2(x)$) then the area in question will be:

$$\int_a^b (f_1(x) - f_2(x))\,dx$$

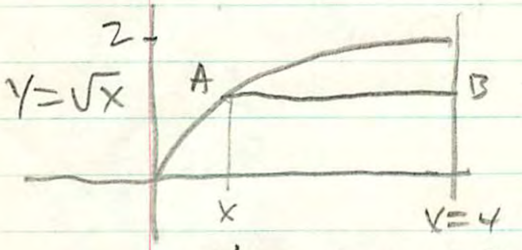
10.1
Pg 362-367

EXAmple 4



$$\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3} - 0 = \frac{16}{3}$$

But there is Nothing Sacred about vertical cross sections.



call c(Y) the cross sections (this time horizontal measurement)

Y varies from 0 to 2

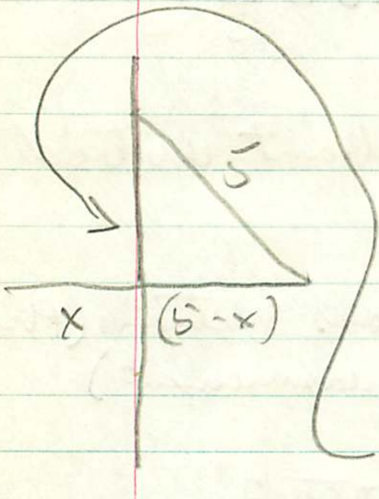
the cross Section is the x coord of B − x coord of A. Since $Y=\sqrt{x}$ then $Y^2 = x$ so the x coord of A is Y^2. Thus the cross Section length will be $4-Y^2$ so we get:

$$\int_0^2 4-Y^2\,dY = 4Y - \frac{1}{3}Y^3\Big|_0^2 =$$

$$8 - \frac{8}{3} = \frac{24-8}{3} = \frac{16}{3}.$$

~~Now Show pblm 15 Pg 366~~

Don't assign 16 and 15 Doesn't need to be done —
nothing gained by the poorly stated problem any way!



$\sqrt{25-(5-x)^2}$?

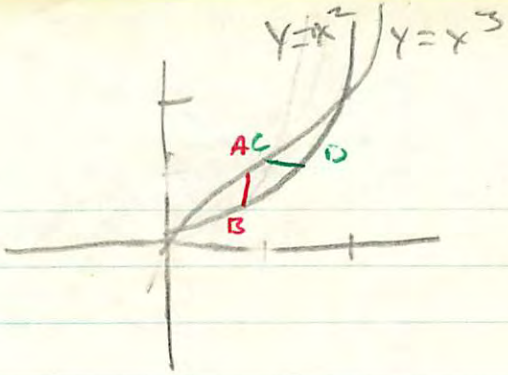
$\sqrt{25-25+10x-x^2}$

$\sqrt{10x-x^2}$

Not sure now.

$$2\cdot 2\int_{\underset{\underset{a}{\uparrow}}{0}}^{b \rightarrow 5}\sqrt{10x-x^2}\,dx$$

①

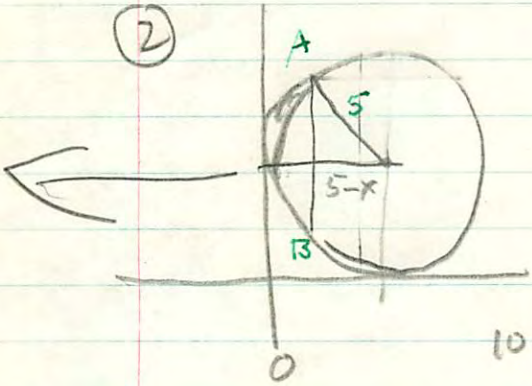


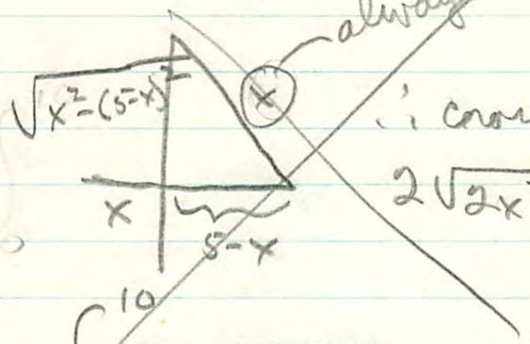
$x^2 - x^3$

$C(x) = \cancel{x^3 - x^2}$

$C(y) = \sqrt[3]{y} - \sqrt[2]{y}$

②





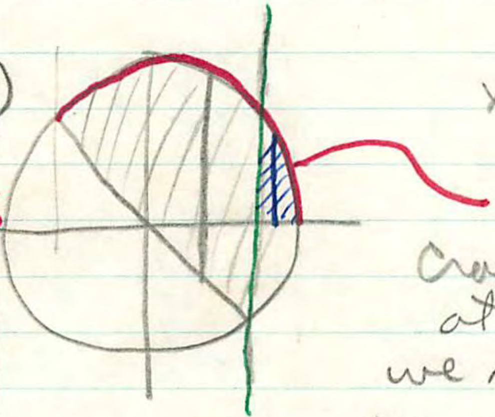
always 5

∴ cross sections $2\sqrt{2x^2+10x-25}$

$$2\int_0^{10} \sqrt{2x^2+10x-25}\,dx$$

Set up integral do not evaluate

③



$x^2+y^2=9 \qquad y=-x$

$y=\sqrt{9-x^2}$

Cross section changes its nature at the green line
we need x values where Both curves intersect

$x^2+x^2=9$
$x^2=\frac{9}{2}$
$x=\pm\frac{3\sqrt{2}}{2}$

So far we have $\displaystyle\int_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} + 2\int_{\frac{3\sqrt{2}}{2}}^{3}$

I have doubled the measure of the Blue shaded area.
so integrals are

$$\int_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} \sqrt{9-x^2}-(-x)\,dx + 2\int_{\frac{3\sqrt{2}}{2}}^{3} \sqrt{9-x^2}\,dx$$

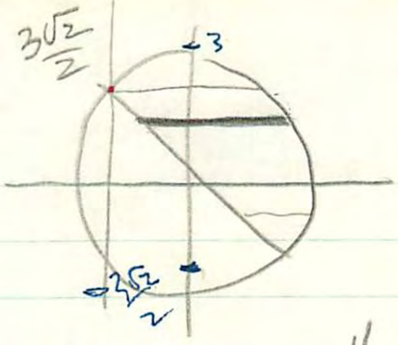
$x^2+y^2=9$ Pg 366

$x=\sqrt{9-y^2}$

$y=-x$

$-y=x$

* ④



$\frac{3\sqrt{2}}{2}$
3
$-\frac{3\sqrt{2}}{2}$

$$\int_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} y+\underline{\sqrt{9-y^2}}\,dy +$$

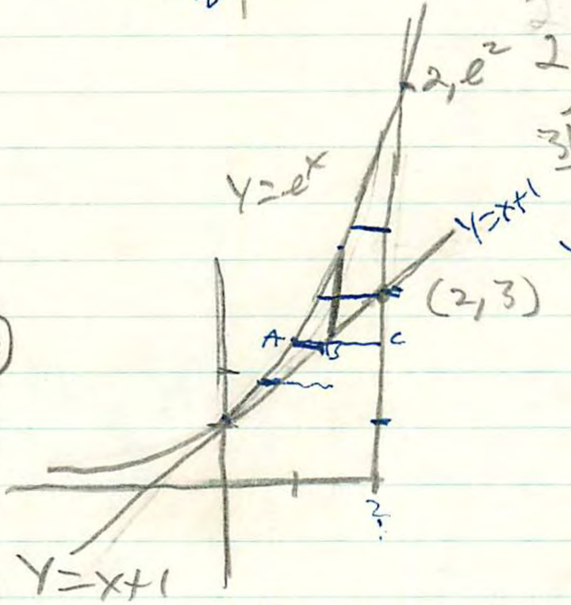
$$2\int_{\frac{3\sqrt{2}}{2}}^{3} \sqrt{9-y^2}\,dy$$

x, e^x

$(x,\ x+1)$

$y-1=x$ $(y-1,\ y)$

* ⑦



2, e^2
$y=e^x$
$y=x+1$
(2, 3)
A
B
C
2
$y=x+1$

$\underline{y-1=x}$

$\underline{\ln y=x}$

$$\int_0^2 e^x-x-1\,dx$$

$2-(y-1)$ y_1

$2-y+1$

$$\int_1^3 y-1-\ln y\,dy +$$

$$\int_3^{e^2} 2-\ln y$$

* ⑧ changes when $y=3$

$$\cancel{\int_1^3 x+1-e^x\,dx + \int_3^{e^2} 2-e^x\,dx}$$

" x value (in terms of y) at B

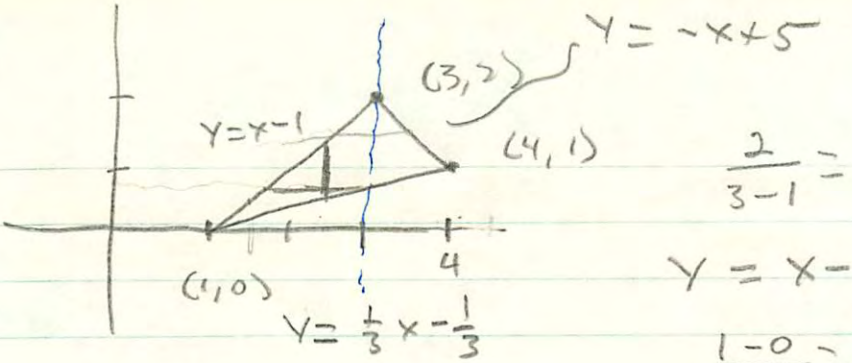
minus

x value (in terms of y) at A

Gives length of $\overline{AB}$ in terms of y.

Pg 366

* ⑨



$\frac{2}{3-1} = 1$

$Y = x-1$

$\frac{1-0}{4-1} = \frac{1}{3}$ $\quad Y = \frac{1}{3}(x-1)$

$Y = \frac{1}{3}x - \frac{1}{3}$

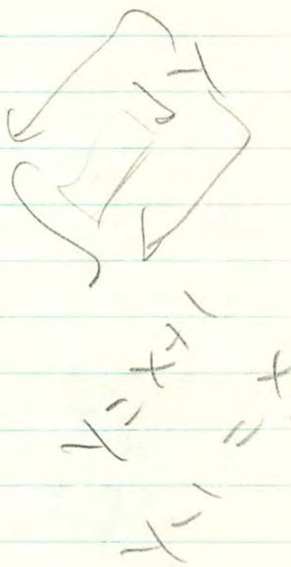
$\frac{2-1}{3-4} = \frac{1}{-1}$ $\quad Y-2 = -x+3$

$Y = -x+5$

$$\int_1^3 (x-1) - \left(\frac{1}{3}x - \frac{1}{3}\right) dx + \int_3^4 (-x+5) - \left(\frac{1}{3}x - \frac{1}{3}\right) dx$$

$$\int_1^3 \frac{2}{3}x - \frac{2}{3}\, dx + \int_3^4 -\frac{4}{3}x + \frac{16}{3}\, dx$$

$$\frac{2}{3}\int_1^3 x-1\, dx + \frac{4}{3}\int_3^4 4-x\, dx$$



* ⑩ $\quad Y = x-1 \qquad Y = -x+5 \qquad 3Y+1 = x$

$Y+1 = x \qquad 5-Y = x$

a)

$$\int_0^1 (3Y+1) - (Y+1)\, dY + \int_1^2 (5-Y) - (Y+1)\, dy$$

$$\int_0^1 2Y\, dy + \int_1^2 4-2Y\, dy$$

b) well both are straight forward, 10a does not involve fractions ∴ it is easier, P, 366

1.8633...

* ⑪ $x^2 = x+1$

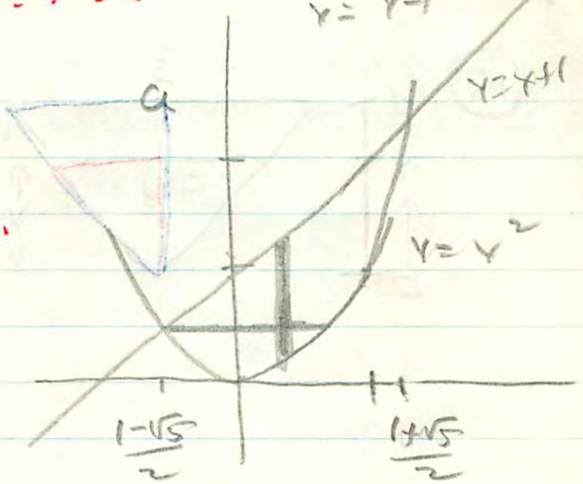
$x^2 - x - 1 = 0$

Integrate using Calculator.

b) $x = \frac{1 \pm \sqrt{1+4}}{2}$

$x = \frac{1 \pm \sqrt{5}}{2}$

$$\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x+1-x^2\,dx$$



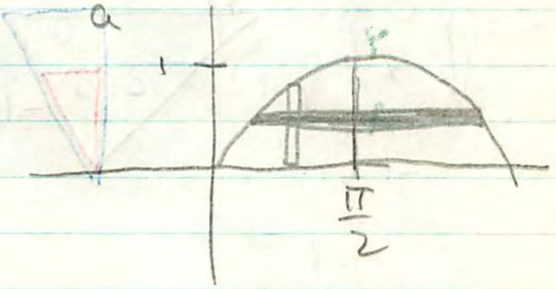
$Y = x+1$

$Y = \frac{1 \pm \sqrt{5}}{2} + \frac{2}{2} = \frac{3 \pm \sqrt{5}}{2}$

c) $$2\int_0^{\frac{3-\sqrt{5}}{2}} \sqrt{Y}\,dy + \int_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} \sqrt{Y} - Y + 1\,dy$$

* ⑫

b) $$\int_0^{\pi} \sin x\,dx$$

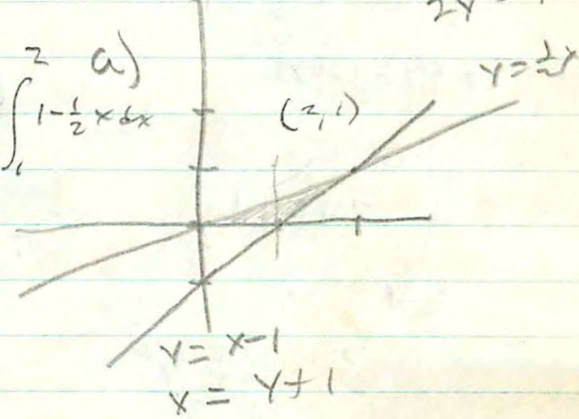


c) $$2\int_0^1 \frac{\pi}{2} - \sin^{-1} y\,dy$$

* ⑬

b $$\int_0^1 \frac{1}{2}x\,dx + \int_1^2 \frac{1}{2}x - x + 1\,dx = \int_0^1 \frac{1}{2}x\,dx + \int_1^2 1 - \frac{1}{2}x\,dx$$

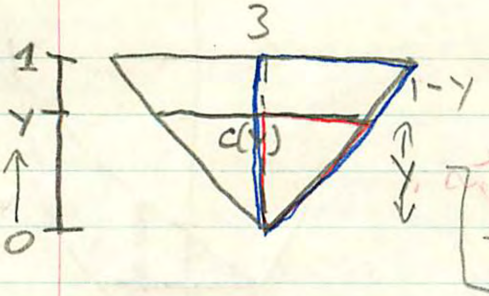
a)



c) $$\int_0^1 Y+1-2Y\,dY = \int_0^1 1-Y\,dy$$

Pg 366

(15)

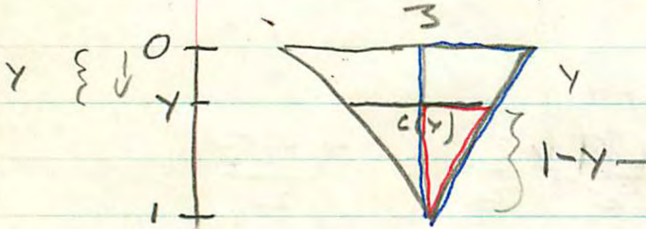


$\frac{3}{1} = \frac{c(y)}{y}$ ∴ $c(y) = 3y$

$\frac{\text{BASE}}{\text{height}} = \frac{\text{Base}}{\text{height}}$

$\int_0^1 3y\,dy$ $\quad \frac{3y^2}{2}\Big|_0^1$

$\frac{3}{2}$

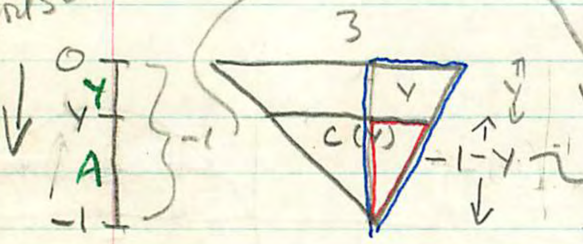


$\frac{3}{1} = \frac{c(y)}{1-y}$ ∴ $c(y) = 3-3y$

$\int_0^1 3-3y\,dy$

$3y - \frac{3}{2}y^2\Big|_0^1 = 3-\frac{3}{2} = \frac{3}{2}$

y starts at 0



$\frac{3}{1} = \frac{c(y)}{1+y}$

$3+3y = c(y)$

~~$A + y = -1$~~
~~$\therefore A = -1-y$~~

$\frac{3}{-1} = \frac{c(y)}{-1-y}$ $\quad \int_0^{-1} 3+3y\,dy =$

$3+3y = c(y)$

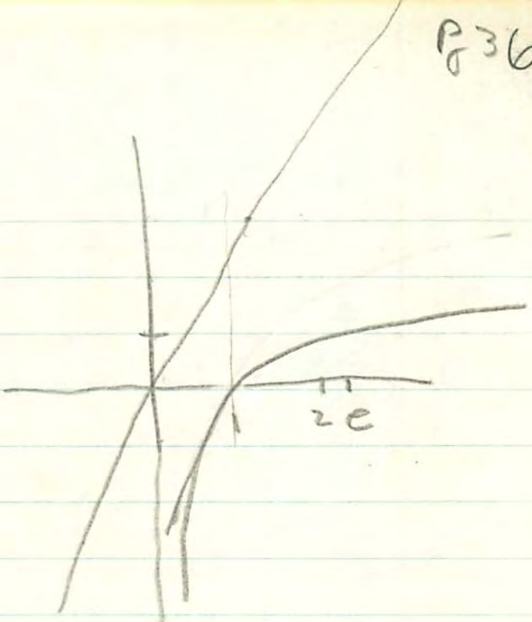
~~Length from 0 to -1 is +1~~
~~y is Negative~~
~~add y to +1 to get~~
~~Actual length of Vertical~~
~~Leg of Δ,~~

$\int_{-1}^{0} 3+3y\,dy$

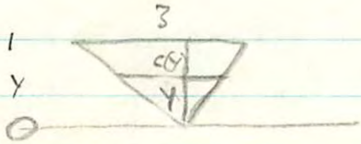
$3y + \frac{3}{2}y^2$

$0 - \{-3+\frac{3}{2}\}$

$\frac{3}{2}$



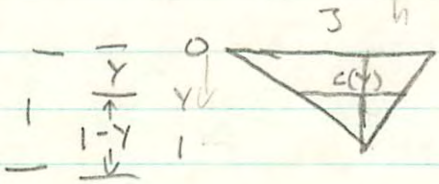
This needs to be included ⑮ in the videos so students can handle #16.



$\int_0^1 3Y\,dy$

$\frac{1}{3} = \frac{Y}{C(Y)}$

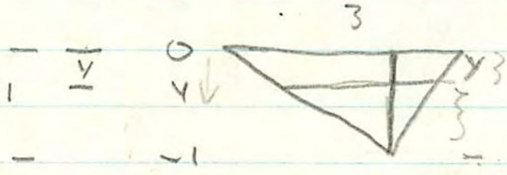
$C(Y) = 3Y$


Y
1
1-Y
0
Y
1
3
h
c(Y)

$\int_0^1 3-3Y$

$\frac{1}{3} = \frac{1-Y}{c(Y)}$

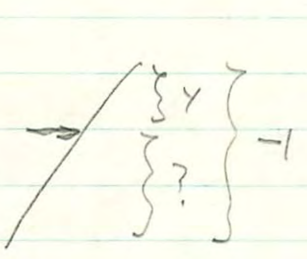
$CY = 3-3Y$



$\int_{-1}^0 3+3Y\,dY$

$\frac{1}{3} = \frac{Y-(-1)}{c(y)}$

$C(-1) = 0$
$C(0) = 3$



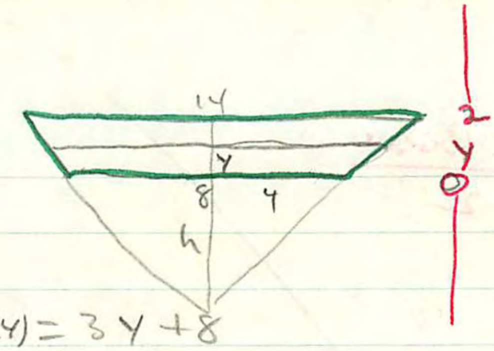
$? + Y = -1$

$? = -Y-1$

$$\frac{3}{1} = \frac{C(Y)}{-Y-1}$$

Pg 367

16A)



$C(y) = 3y + 8$

$C(0) = 8$ $(0,8)$

$C(2) = 14$ $2,14$

$\frac{14-8}{2-0} = 3$

B) making the figure into a triangle shape

$\frac{h}{4} = \frac{h+2}{7}$

$7h = 4h+8$

$3h = 8$

$h = \frac{8}{3}$

$\frac{h}{4} = \frac{h+y}{\frac{1}{2}C(y)}$

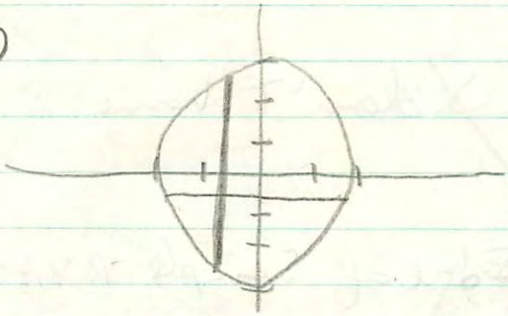
$\frac{1}{2}hC(y) = 4h + 4y$

$\frac{1}{2}\cdot\frac{8}{3}C(y) = 4\left(\frac{8}{3}\right) + 4y$

$C(y) = \frac{3}{4}\left(\frac{32}{3} + 4y\right)$

$C(y) = 8 + 3y$

? ⑰



$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$y = \sqrt{36 - x^2}$

a) $2\int_{-2}^{2} \sqrt{36-x^2}\, dx$

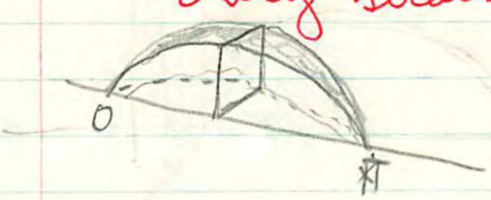
b) $2\int_{-3}^{3} \sqrt{36-y^2}\, dy$

10.2 Using cross sectional area to compute volume

10,2 - 427
Pg 367-371

Just as we used length to give us area we can use area to give us volume.

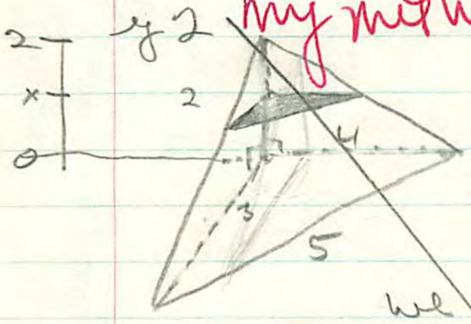
eg 1 Pg 368 we want to set up a definite integral for the volume of a solid whose base is the region bounded by the x axis and the arch y = sin x from 0 to π. ~~the arch is then tagged back until it lies in the plane x~~ See the figure in the text. (The cross sectional area is a square w/ side sin x. We know this only because it was given!



∴ The Area of the cross section is $\sin^2 x$

$$\int^{\pi} \sin^2 x \, dx \; . \quad \left(2\int_0^{\pi/2} \sin^2 x \, dx = \frac{\pi}{2}\right)$$

eg 2 my method is better — On next Page



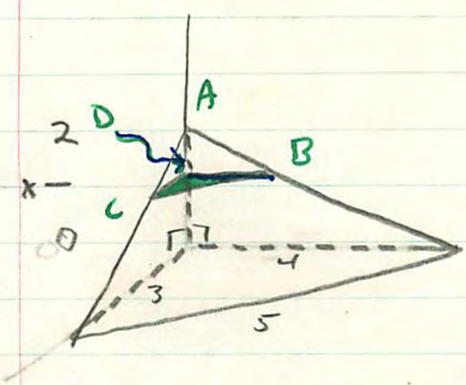
~~Since there are 3 rt △s, we can choose appropriate cross sections to be a plane parallel to one of these 3 sides.~~

~~we choose the vertical Portion of our Pyramid (length 2) to be Parallel to the ~~our~~ x axis.~~

~~Our Sample △ is Similar to the 3-4-5 Base △~~ w/ propality K so ~~the areas have propationality K², if corresponding sides of similar figures have ratio K then the figures have area which has the ratio K². Our sample △, 'T', has dimensions which are Proportional to the Location of x on the axis, we can Say that this distance is the from the Apex to 'T'. in this case that distance is 2-x. thus A(x) is Proportional to the Sq of that distance ie A(x) = K(2-x)² thus when x is zero we have the Base Triangle which has area ½·3·4 = 6 so~~

10.2
Pg 367-371

Example #2



$\overline{AD} = 2-x$

BD: $\frac{BD}{AD} = \frac{4}{2} \therefore BD = 2AD$

$BD = 2(2-x)$

CD: $\frac{CD}{AD} = \frac{3}{2} \therefore CD = \frac{3}{2}(2-x)$

Area of $\triangle BCD = \frac{1}{2}\overline{CD} \cdot \overline{BD} =$

$\frac{1}{\cancel{2}} \cdot \cancel{2}(2-x) \cdot \frac{3}{2} \cdot (2-x) =$

$\frac{3}{2}(2-x)^2$

Since $\triangle BCD$ is a "typical" cross section

Each triangle has this area. So volume of pyramid is $\int_0^2 \frac{3}{2}(2-x)^2 dx$.

10.2

pg 367-371

~~$A(0) = K(2-0)^2 = 6$~~

~~$K = \frac{3}{2}$ the proportionality const~~ ? Uuh — some very obscure Geo property?

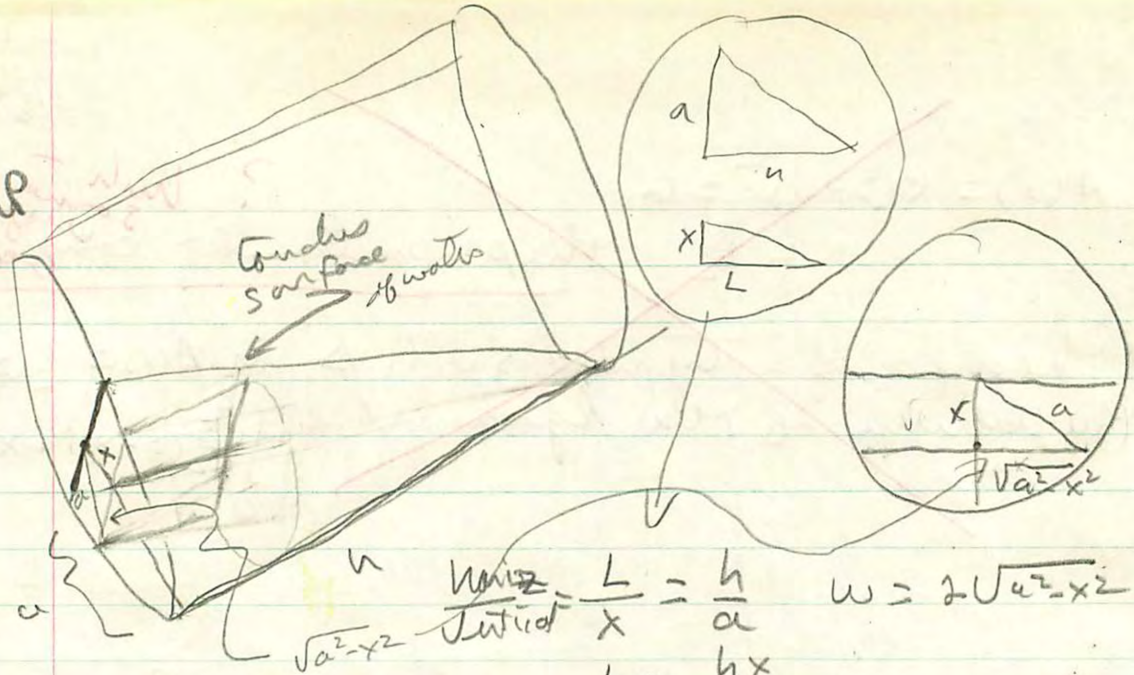
~~The area of a typical such Δ is $A(x) = \frac{3}{2}(2-x)^2$~~
~~the Volume of the Pyramid is $\int_0^2 \frac{3}{2}(2-x)^2 dx$~~

eg 3 we have done Similar problems in chapt 7
The only difficult part is representing the radius of the Typical circle in terms of a and x,

eg 4 is a classic integration problem which every class struggles through. a lot can be learned from this so we dive in for what we can learn and not for tradition.

a drinking glass is in the form of a right circular cylinder with radius "a" and hieght h. The glass is to be tilted until the water line bisects the base and touches the rim at the same time,
The Job is to find the area of a typical cross section so we could compute the Volume of water still in the glass

Following the discussion in the text begining on pg 370 method #4 Set up



Touches surface of water
a
h
x
a
$\sqrt{a^2-x^2}$
a
h
x
L
x
a
$\sqrt{a^2-x^2}$

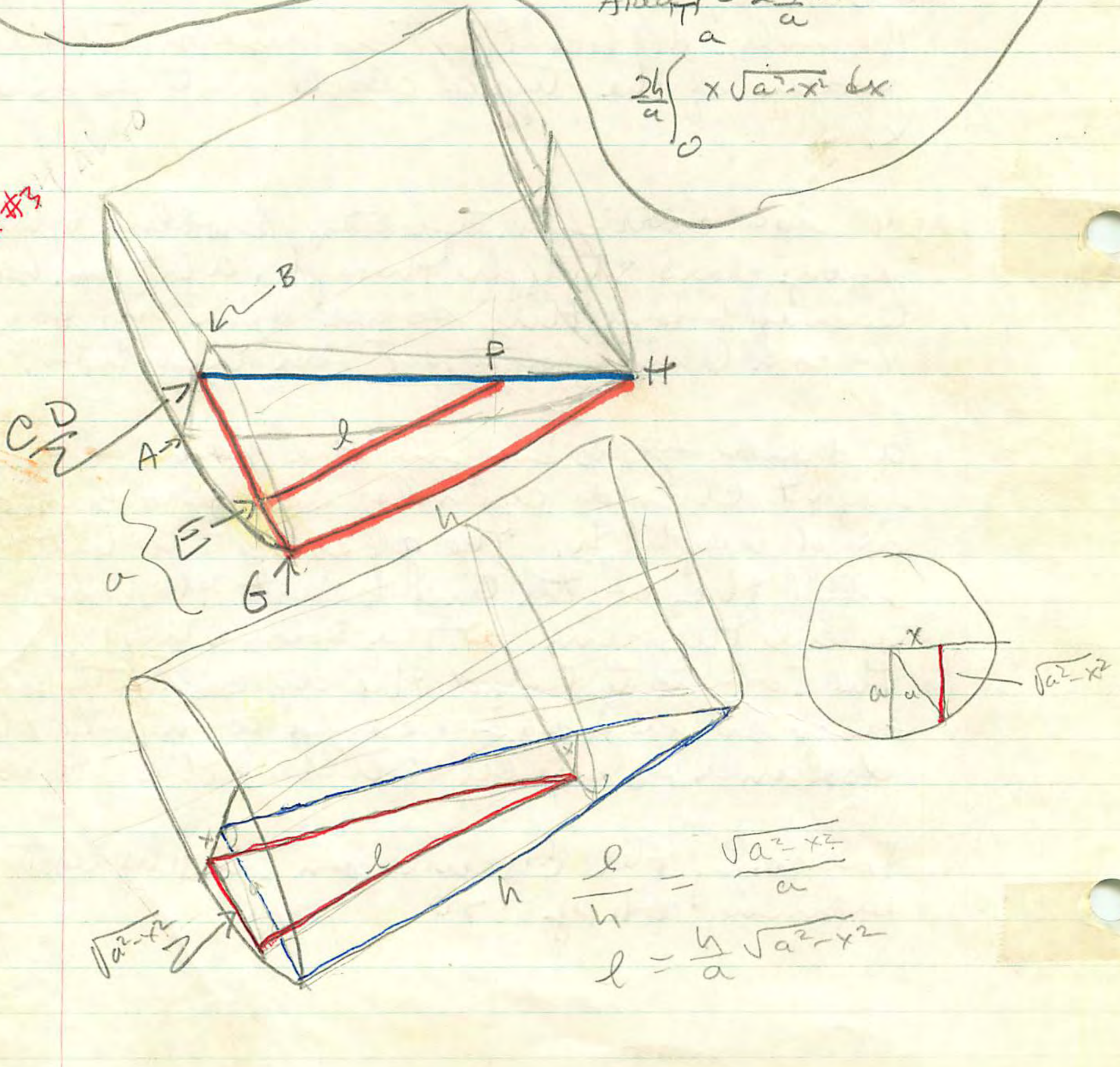
$$\frac{\text{horiz}}{\text{vertical}} = \frac{L}{x} = \frac{h}{a} \qquad w = 2\sqrt{a^2-x^2}$$

$$L = \frac{hx}{a}$$

$$\text{Area}_T = 2\frac{h}{a}x\sqrt{a^2-x^2}$$

$$\frac{2h}{a}\int_0^a x\sqrt{a^2-x^2}\,dx$$

method #3



B
P
H
C D
A
l
E
a
G
h
x
l
h
$\sqrt{a^2-x^2}$
x
a
a
$\sqrt{a^2-x^2}$

$$\frac{l}{h} = \frac{\sqrt{a^2-x^2}}{a}$$

$$l = \frac{h}{a}\sqrt{a^2-x^2}$$

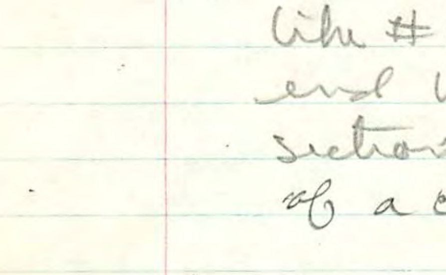
10.2
Pg 367-371

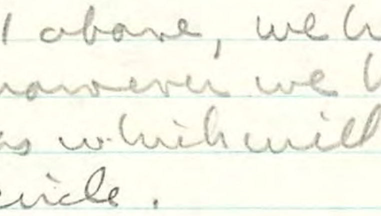
We are presented with 4 methods of attack.

#1. work with Planes parallel to the surface of the water. This is impractical since the surface of water is 1/2 of an ellipse. while this isn't too bad, the other "typical" cross sections will not be convenient portions of an ellipse. (area of ellipse is πab)

#2 work w/ Planes parallel to the base of the Glass, like #1 above, we have 1/2 of a circle at one end however we have other typical cross sections which will not be convenient portions of a circle.

#3 Planes ⊥ to AB (the Diameter). This method has some nice features. Each cross section will be a rt. Δ.
1st we will place an x axis parallel to the base and locate the origin O to correspond to center of the base



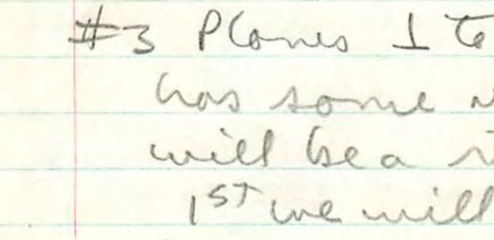


The Area of the typical Δ 'T' is $\frac{1}{2}\overline{DE}\cdot\overline{EF}$. (This can be seen easily using any reasonable glass and looking)

1st we consider $\overline{DE}$. $\overline{CD}=x$, $\overline{CE}=a$ (the radius) ∴ $DE\ \sqrt{a^2-x^2}$. Now EF: $GH=h$ (height) $\overline{CG}=a$.

$$\frac{DE}{CG}=\frac{EF}{GH} \Rightarrow \frac{\sqrt{a^2-x^2}}{a}=\frac{EF}{h} \quad \text{so}$$

$$EF=\frac{h}{a}\sqrt{a^2-x^2}$$



C, D appear at same pt. actually C is behind D.

10.2
Pg 367-371

$\frac{1}{2} l w$

$\frac{1}{2} \frac{h}{a} \sqrt{a^2-x^2} \cdot \sqrt{a^2-x^2}$

Thus area of 'T' is $\frac{1}{2} \cdot \frac{h}{a} \left(\sqrt{a^2-x^2}\right)^2 =$

$\frac{h}{2a}(a^2-x^2)$ Thus the volume of the water will be given by $\int_{-a}^{a} \frac{h}{2a}(a^2-x^2)$

method #4 using Planes ⊥ to base and Parallel to the bottom edge of the glass, a typical cross sections is a rectangle. Since the area of a rectangle is easy to compute, this is another method. (This is pblm 9 pg 372)

$$\frac{h}{2a}\int_{-a}^{a} a^2 - x^2\,dx = \frac{h}{2a}\left(a^2x \cdot \frac{x^3}{3}\right)\Big|_{-a}^{a}$$

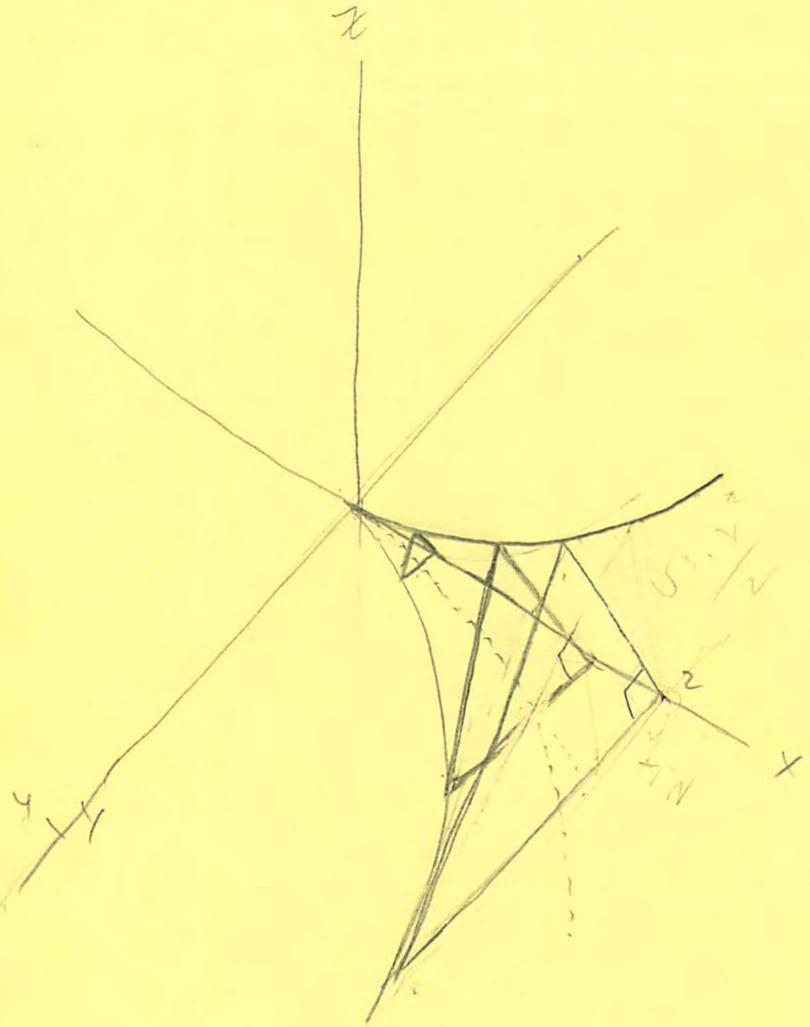
$$\frac{h}{2a}\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$\frac{h}{2a}\left[2a^3 - \frac{2a^3}{3}\right] = \frac{h}{\cancel{2}a}\left[\frac{\overset{2}{\cancel{4}}a^3}{3}\right] = \frac{2}{3}a^2h$$ water remaining

full glass $\pi r^2 h \Rightarrow \pi a^2 h$

$$\frac{\frac{2a^2h}{3}}{\pi a^2 h} = \frac{2\cancel{a^2h}}{3\pi \cancel{a^2h}} \quad \frac{2}{3\pi} \approx \frac{2}{3.3.1} = \frac{2}{9.3} \approx \frac{1}{5}$$

of water remains.

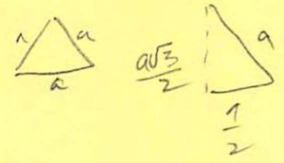


#2 pg 372

w/out a pict....
what do we know?

equilateral Δ.

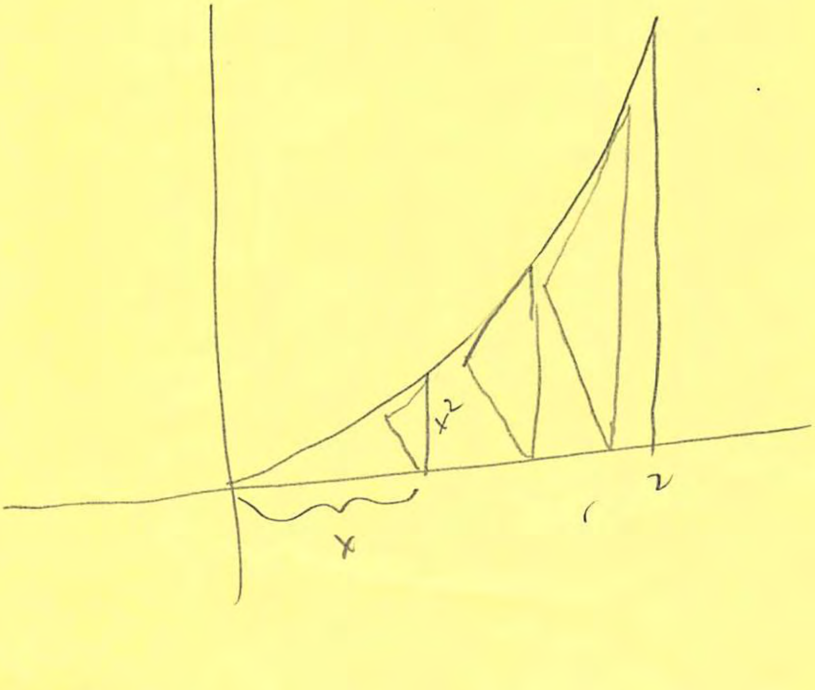
So

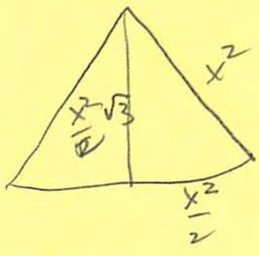

a
a
a
$\frac{a\sqrt{3}}{2}$
a
$\frac{1}{2}$

all sides = ...
. one side x^2 . . .

So $A_{max} = \frac{1}{2} x^2 \cdot \frac{x^2\sqrt{3}}{2}$

now show pict.




x^2
$\frac{x^2\sqrt{3}}{2}$
$\frac{x^2}{2}$

$$\int_0^2 \frac{1}{2} x^2 \cdot \frac{x^2\sqrt{3}}{2}\, dx$$

Pg 371-372

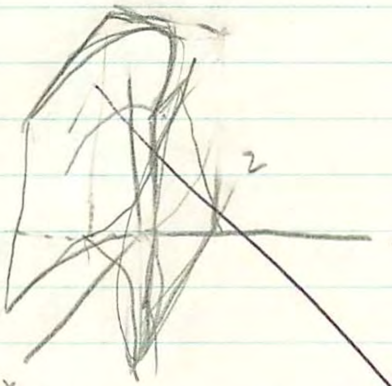
① a) $A = s^2$ $A = K^2 s$

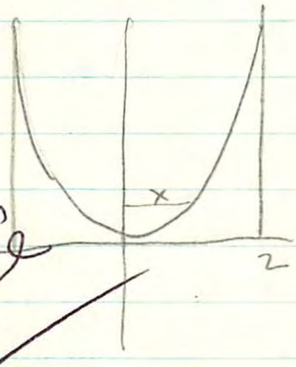
b) $\frac{K^2 s^2}{s^2} = K^2$

c) This illustrates the statement at the top of pg 368 ie if corresponding sides have proportionality K their areas have proportionality K^2.

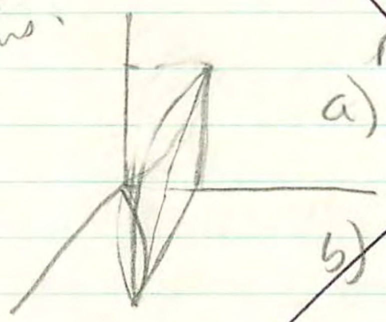
?o, ②

No Not Δ on cross sections.





See previous page



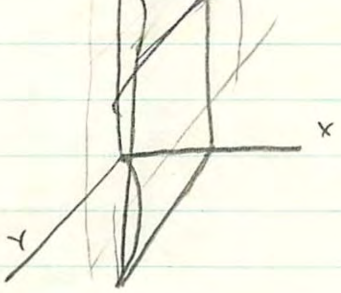
~~Pivot $y = x^2$ at origin up to~~

~~a) xz plane to form solid~~

~~b) Typical cross section is Equilateral Δ?~~

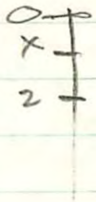
Maybe a washer

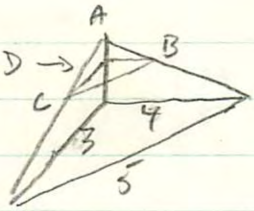
circle rad 2 — πx^2



p 372
6

#3





$AD = x$

$\frac{BD}{AD} = \frac{4}{2} \therefore BD = 2x$

$\frac{CD}{AD} = \frac{3}{2} \therefore CD = \frac{3}{2}x$

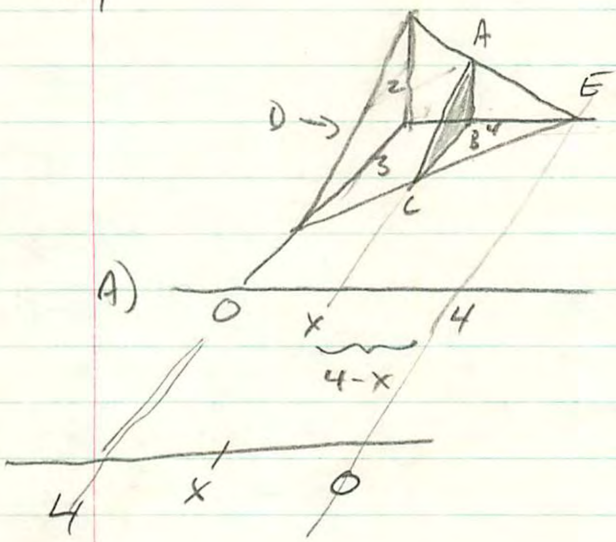
$A = \frac{1}{2}\,\overline{BD}\;\overline{CD}$

$= \frac{1}{2} \cdot 2x \cdot \frac{3}{2}x$

$= \frac{3}{2}x^2$

$\int_0^2 \frac{3}{2}x^2\,dx$

#4



$BE = 4-x$

$\frac{AB}{BE} = \frac{2}{4}$ $\triangle ABE \angle$

$AB = \frac{1}{2}(4-x)$

$\frac{BC}{BE} = \frac{3}{4}$

$BC = \frac{3}{4}(4-x)$

$\int_0^4 \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4}(4-x)^2 = \frac{3}{16}\int (4-x)^2\,dx$

B) $BE = x$

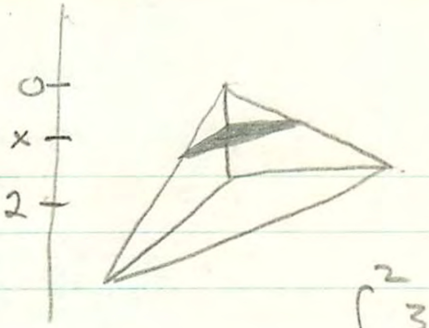
$AB = \frac{1}{2}x$

$BC = \frac{3}{4}x$

$\frac{3}{16}\int_0^4 x^2\,dx$

Pg 372

③



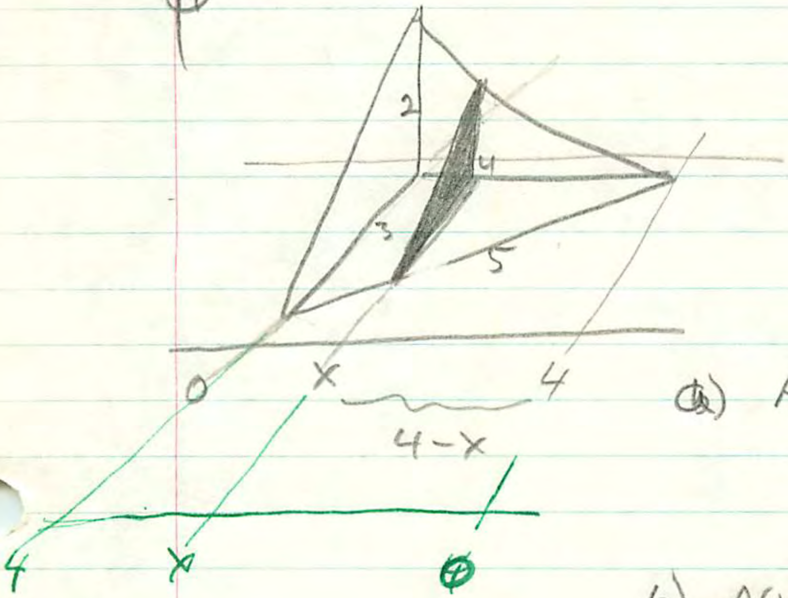
$A(x) = Kx^2$

$A(2) = 6 \therefore A(2) = K(2)^2 = 6$

$K = \frac{3}{2}$

$\int_0^2 \frac{3}{2}x^2\,dx$

④



$A(x) = K(4-x)^2$

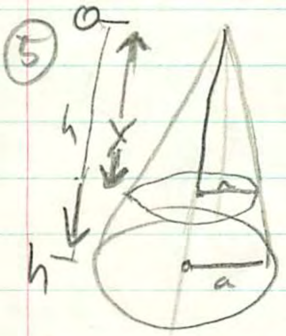
$A(0) = 3 = K(4)^2$

$\frac{3}{16} = K$

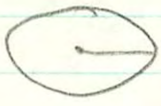
a) $A(x) = \frac{3}{16}(4-x)^2$ $\frac{3}{16}\int (4-x)^2\,dx$

b) $A(x) = \frac{3}{16}x^2$ $\frac{3}{16}\int x^2\,dx$

⑤



$A(x) = \pi r^2$



$A(x) = \pi \frac{a^2x^2}{h^2}$

$A(1) = \pi a^2 \therefore K = \pi$

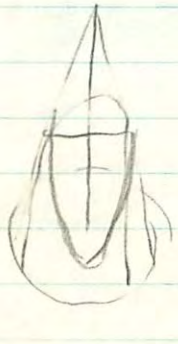
$\frac{x}{r} = \frac{h}{a}$

$\frac{ax}{h} = r$

$\int_0^h \pi \frac{a^2x^2}{h^2}\,da$

$A = \pi\left(\frac{ax}{h}\right)^2$

⑥

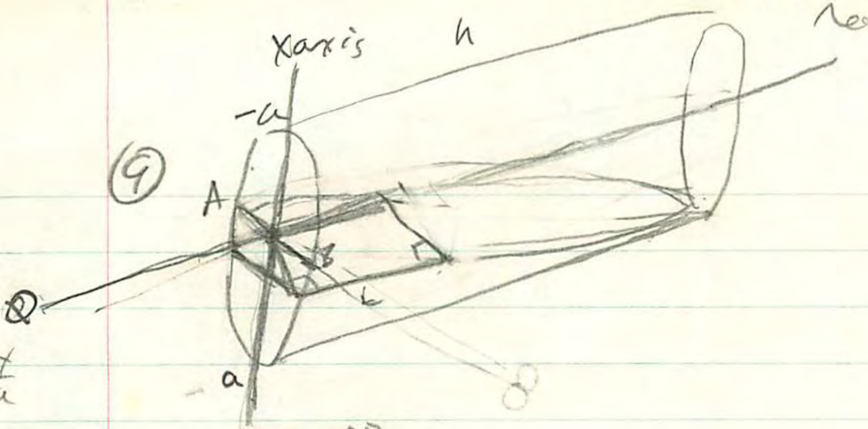


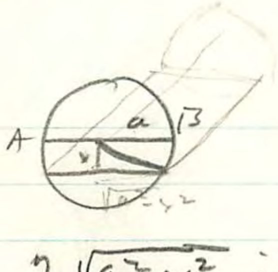
Part of hyperbola.

radius a

Pg 372

⑨


X axis
h
-a
A
B
L
a


A
a
B
x
$\sqrt{a^2-x^2}$

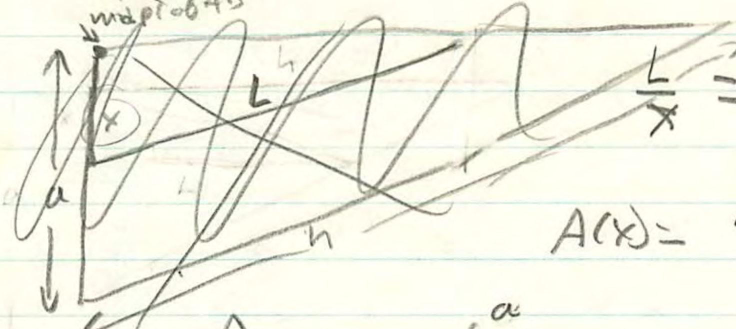
$2\sqrt{a^2-x^2}$ is width of rectangle.

$L = \frac{hx}{a}$

$\frac{L}{h} = \frac{x}{a}$

$\frac{a}{h} = \frac{x}{L}$

$\frac{h}{a} = \frac{L}{x}$


midpt of AB
x
L
h
a
h

$\frac{L}{x} = \frac{h}{a} \Rightarrow L = \frac{hx}{a}$ b)

$A(x) = \frac{2hx}{a}\sqrt{a^2-x^2}$ c)

$$\frac{2h}{a}\int_0^a x\sqrt{a^2-x^2}\,dx$$

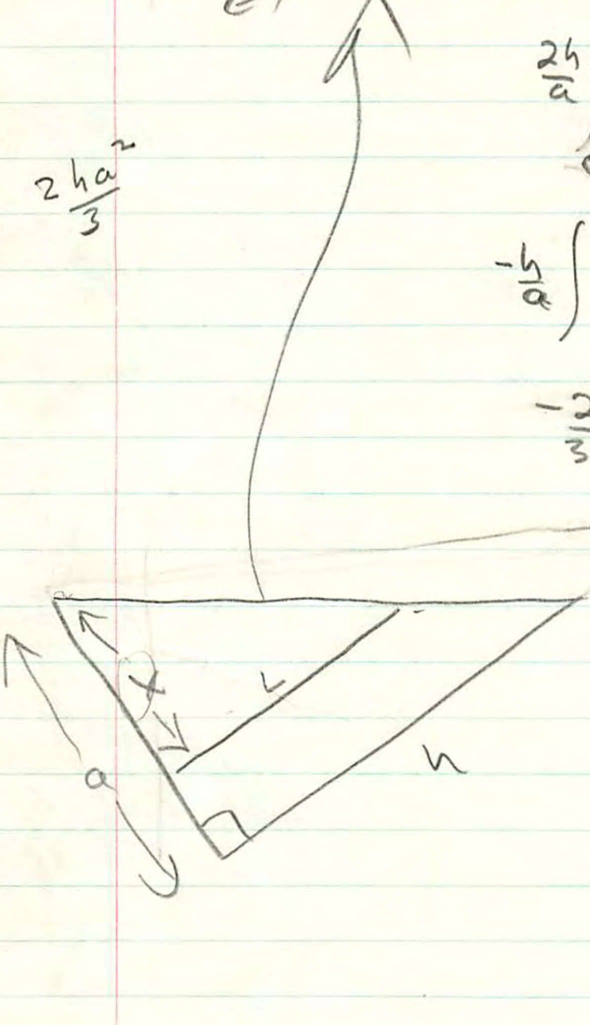
let $u = a^2-x^2$
$du = -2x\,dx$

$\frac{2ha^2}{3}$

$$-\frac{h}{a}\int u^{\frac{1}{2}}\,du = -\frac{h}{a}\frac{2}{3}u^{3/2} =$$

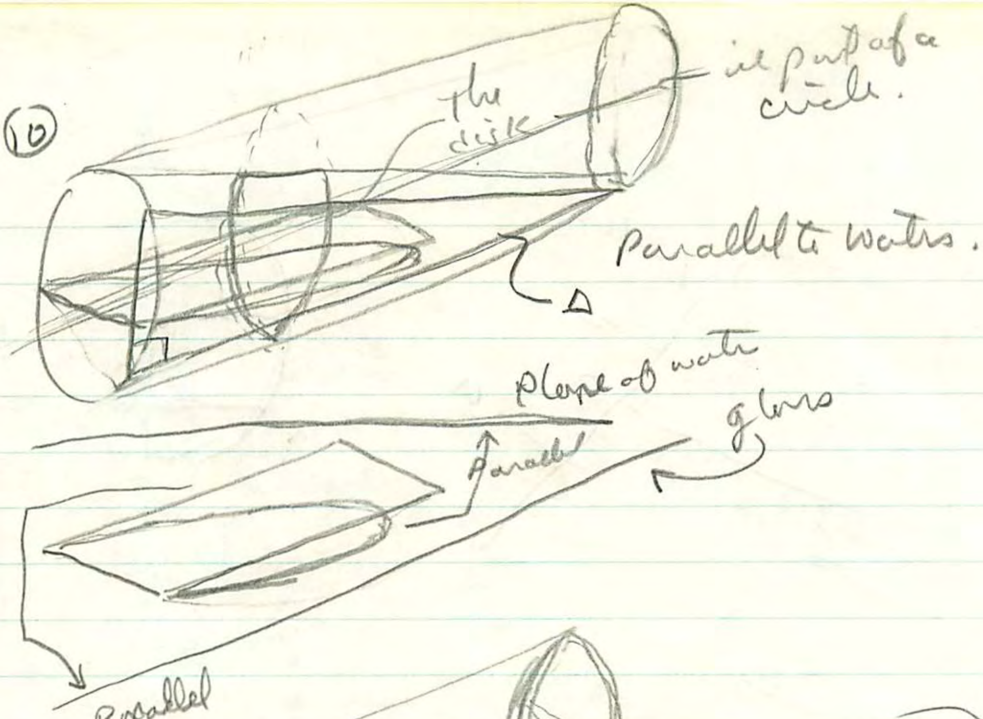
$$-\frac{2h}{3a}(a^2-x^2)\sqrt{a^2-x^2}\Big|_0^a = 0-\left(\frac{-2h}{3\cancel{a}}(a^2)(\cancel{a})\right)$$

So volume is, $\frac{2h\,a^2}{3}$ ans.

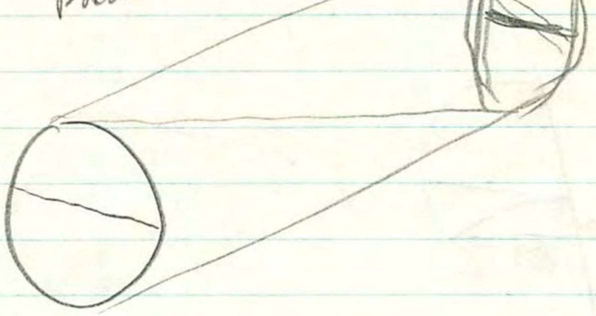


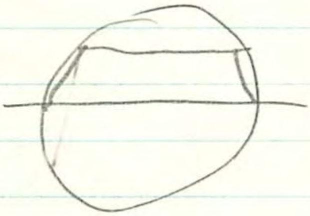
p 332

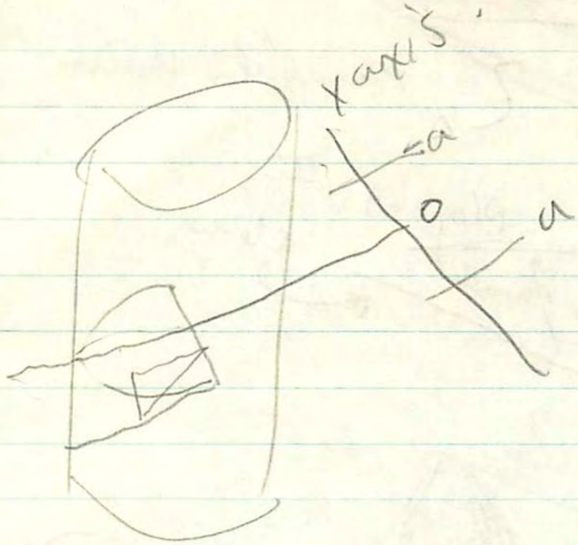
(10)

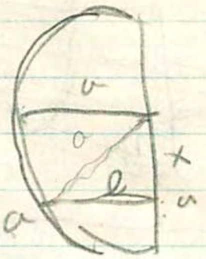

the disk
ie part of a circle.
Parallel to waters.
Δ
plane of water
glass
parallel
Parallel

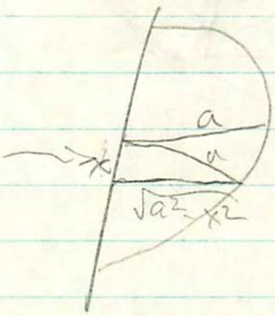
(11)







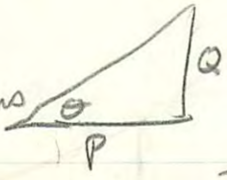



a
a
x
$\sqrt{a^2-x^2}$

$$\cos\theta = \frac{\overset{?}{1}}{\sqrt{a^2-x^2}}$$

Pg 372

Remember θ is a constant in these problems

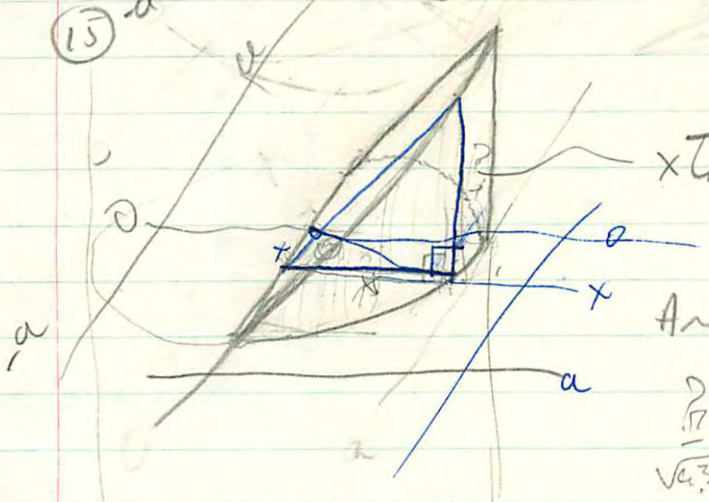


$\tan\theta = \frac{Q}{P}$ ∴

$Q = P\tan\theta$

(14)

(15)


a
θ
x tan θ
θ
x
x
a
a

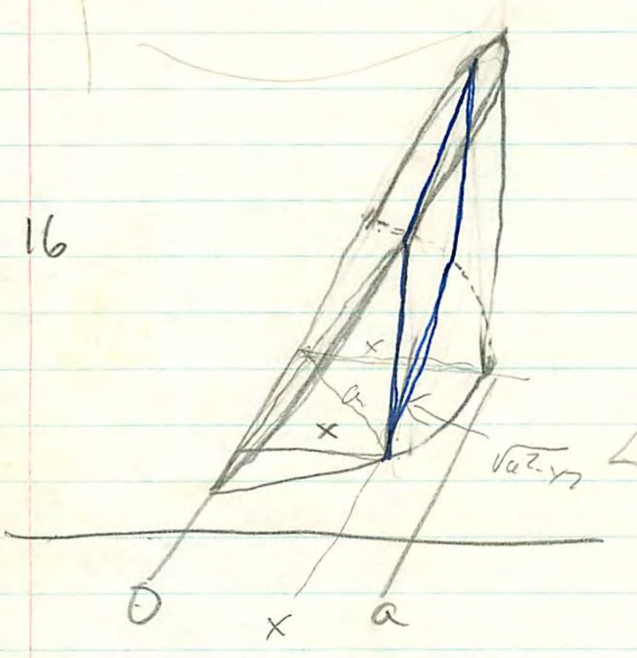
Area of typical ND Δ is $\frac{1}{2}x \cdot x\tan\theta$

$\frac{x^2}{2}\tan\theta$.

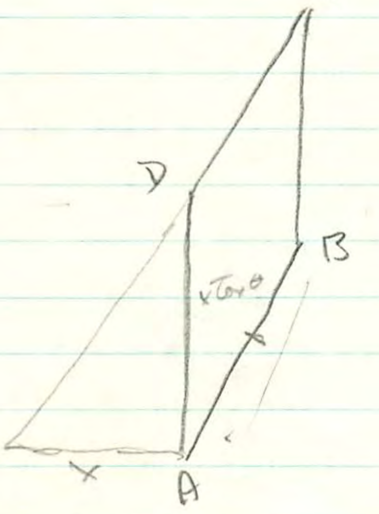
$\frac{?}{\sqrt{a^2-x^2}} = \tan\theta$

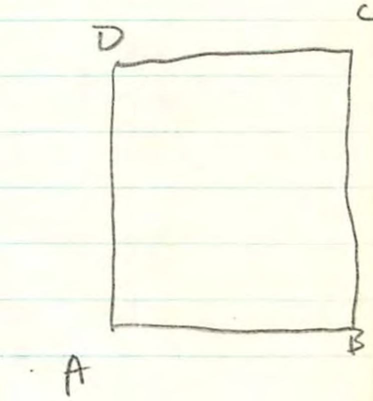
⟵ $\frac{1}{2}(a^2-x^2)\tan\theta$

16


x
a
x
$\sqrt{a^2-x^2}$
O
x
a

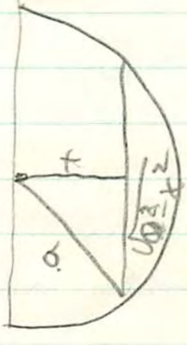
Typical


C
D
B
x tan θ
x
x
A

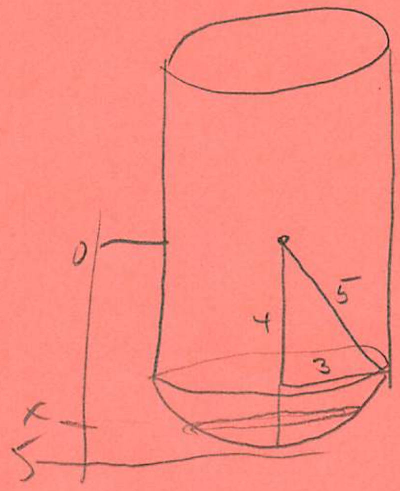


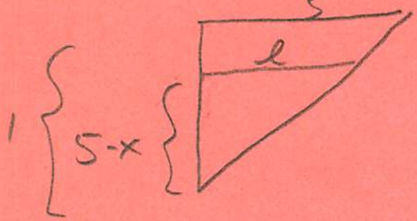
w/out pict...

$2x\sqrt{a^2-x^2}\tan\theta$


x
a
$\sqrt{a^2-x^2}$

#18
Pg 372





$$\frac{l}{3} = \frac{5-x}{1}$$

$$l = 3(5-x)$$

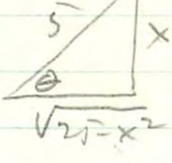
$$2\left[\pi\int_4^5 \left[\boxed{3}(5-x)\right]^2 dx + \int_0^4 3^2\pi \, dx \right]$$

Pg 323

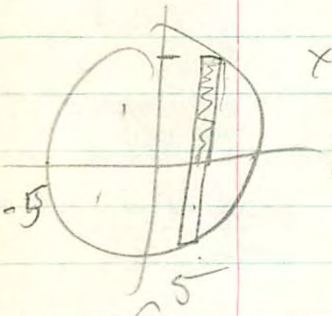
① $\int_{-5}^{5} 2\sqrt{25-x^2}\,dx$ let $x^2 = 25\sin^2\theta$

$x = 5\sin\theta$

$dx = 5\cos\theta\,d\theta$



$x^2+y^2=25$ $\quad y = \sqrt{25-x^2}$



$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 25\cos^2\theta\,d\theta$

$x=-5 \Rightarrow \theta = \frac{3\pi}{2} = -\frac{\pi}{2}$

$x=5 \Rightarrow \theta = \frac{\pi}{2}$

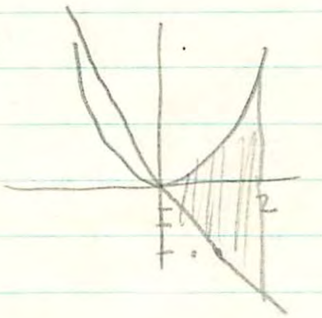
$4\int_{0}^{5}$

$50\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,d\theta = 25\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta\,d\theta\right]$

$25\,\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{25}{2}\sin 2\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$

$\frac{25\pi}{2} + \frac{25\pi}{2} + 0 \quad = 25\pi$

②

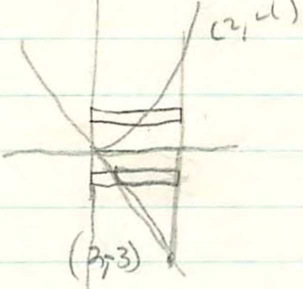


$\int_0^2 x^2 + \frac{3}{2}x\,dx = \frac{1}{3}x^3 + \frac{3}{4}x^2\Big|_0^2$

$\frac{8}{3} + 3 = \frac{17}{3}$

③

$\frac{1}{2}x^2, 3$



$y = x^2 \quad y = -\frac{3}{2}x \quad x = 2$

$\sqrt{y} = x \quad -\frac{2}{3}y = x$

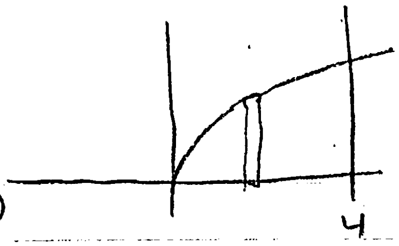
$\int_{-3}^{0} 2 + \frac{2}{3}y\,dy + \int_0^4 2 - \sqrt{y}\,dy$

$2y + \frac{1}{3}y^2\Big|_{-3}^{0} + 2y - \frac{2}{3}y^{3/2}\Big|_0^4$

$-(-6+3) + 8 - \frac{2}{3}\cdot 8$

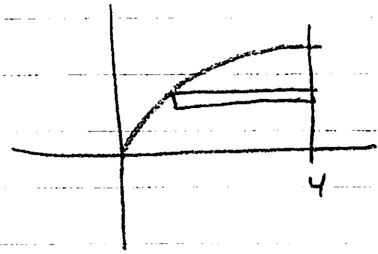
$3 + 8 - \frac{16}{3} \quad \frac{33}{3} - \frac{16}{3} = \frac{17}{3}$

④ a)



$\int_0^4 x^{\frac{1}{2}}dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}$

b)



$\int_0^2 4-y^2\,dy = 4y-\frac{1}{3}y^3\Big|_0^2 = 8-\frac{8}{3} = \frac{16}{3}$

⑤ $\int_0^2 e^x-x-1\,dx = e^x-\frac{1}{2}x^2-x\Big|_0^2 = e^2-2-2-1$

e^2-5

Skip ⑥

⑦ Too messy

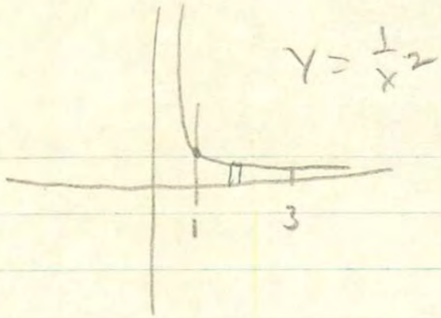
⑧ a) $\int_0^1 \frac{1}{2}x\,dx + \int_1^2 1-\frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_0^1 + x-\frac{1}{4}x^2\Big|_1^2$

$\frac{1}{4}+2-1-1+\frac{1}{4} = \frac{1}{2}$

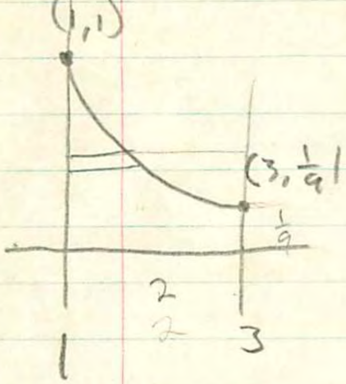
b) $\int_0^1 1-y\,dy = y-\frac{1}{2}y^2\Big|_0^1 = 1-\frac{1}{2} = \frac{1}{2}$

P. 373

⑪



$$\int_1^3 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^3 = -\frac{1}{3}+1 = \frac{2}{3}$$



$$\frac{1}{y} = x^2$$

$$\frac{1}{\sqrt{y}} = x$$

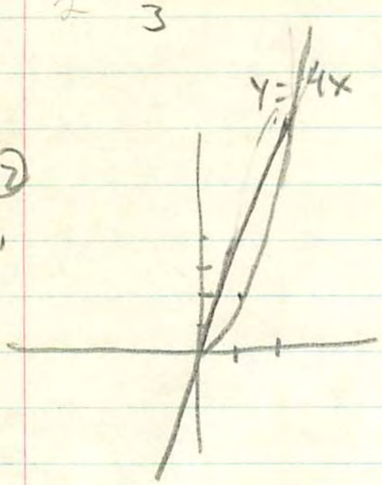
$$\int_0^{\frac{1}{9}} 2\,dy + \int_{\frac{1}{9}}^1 \frac{1}{\sqrt{y}} - 1\,dy$$

$$2y\Big|_0^{\frac{1}{9}} + 2\sqrt{y} - y\Big|_{\frac{1}{9}}^1 \quad \frac{2}{9} + 2 - 1 - \frac{2}{3} + \frac{1}{9}$$

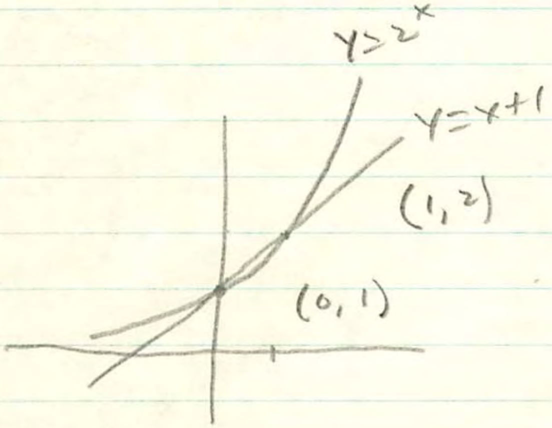
$$1 - \frac{1}{3} = \frac{2}{3}$$

⑫

boring,



⑬



$$\int_0^1 x + 1 - 2^x\,dx =$$

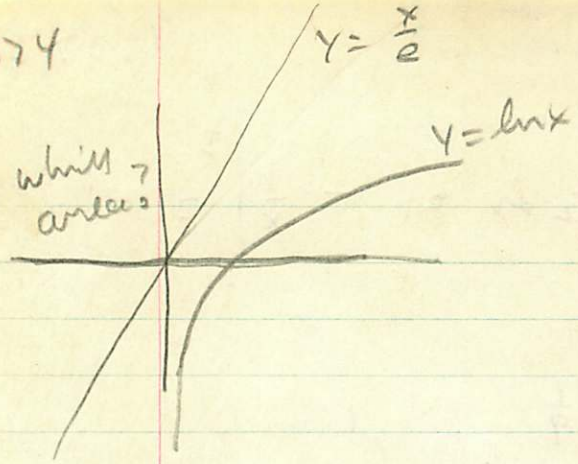
$$\frac{1}{2}x^2 + x - \frac{2^x}{\ln 2}\Big|_0^1 = \frac{1}{2} + 1 - \frac{2}{\ln 2} + \frac{1}{\ln 2}$$

$$\int 2^x\,dx = \int e^{\ln 2^x}\,dx = \int e^{x\ln 2}\,dx = \frac{1}{\ln 2}e^{x\ln 2} = \frac{2^x}{\ln 2}$$

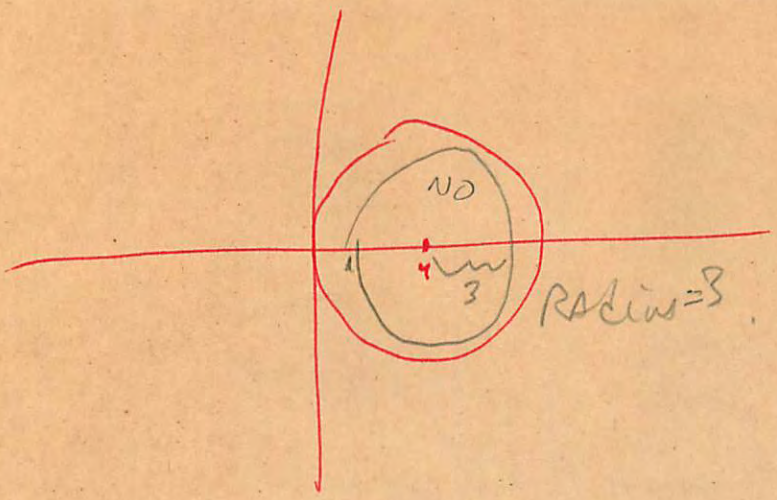
$$\frac{3}{2} - \frac{1}{\ln 2}$$

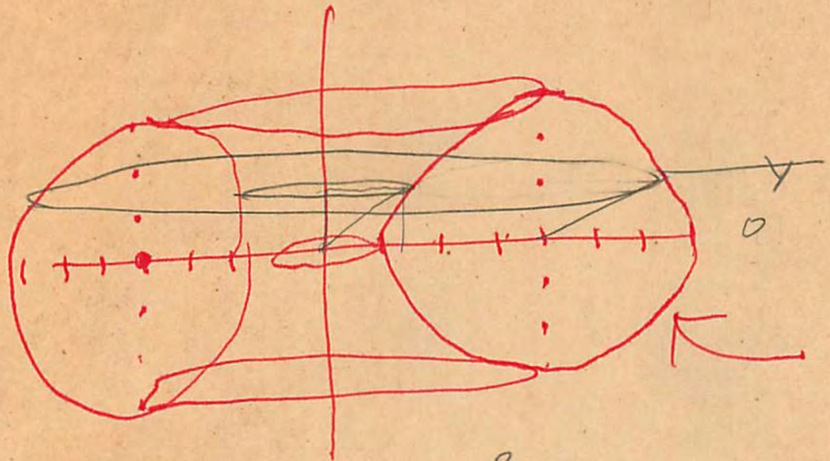
Pg 374

(14) which area?


$y = \frac{x}{e}$
$y = \ln x$

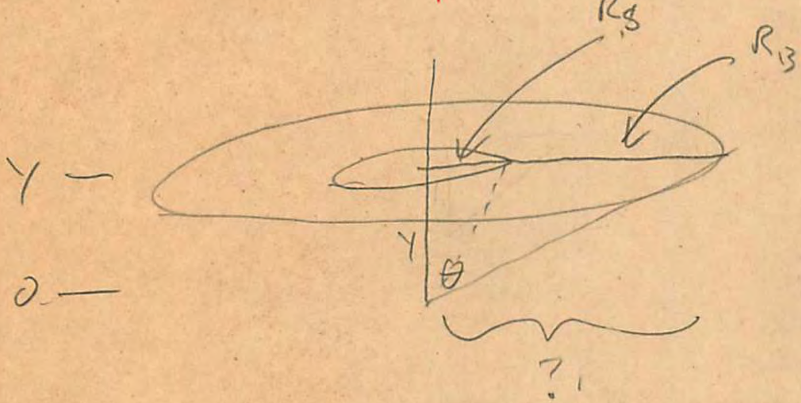
#17


NO
1
4
3
RAdius=3



$(x-4)^2+y^2=9$

$x=4\pm\sqrt{9-y^2}$



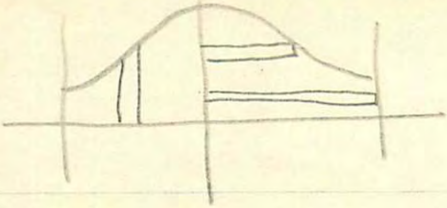
$\frac{R_B}{Y}=\tan\theta.$

$R_B=4+\sqrt{9-y^2}$

and

$R_S=4-\sqrt{9-y^2}$

Pg 374



⑲ $Y = \frac{1}{1+x^2}$

Vertical cross sections

$2\int_0^1 \frac{1}{1+x^2}\,dx = 2\tan^{-1}x\Big|_0^1 = 2\tan^{-1}1 - 2\tan^{-1}0 = 2\cdot\frac{\pi}{4} - 2\cdot 0 = \frac{\pi}{2}$

$Y + Yx^2 = 1$

$x^2 = \frac{1-Y}{Y}$

$x = \sqrt{\frac{1-Y}{Y}}$

$2\int_0^{\frac{1}{2}} dy + 2\int_{\frac{1}{2}}^1 \sqrt{\frac{1-Y}{Y}}\,dy$

Let $u = \sqrt{Y}$

$u^2 = Y$

$2u\,du = dY$

$x = \frac{1}{2}$ $u = \frac{\sqrt{2}}{2}$

$Y = 1$ $u = 1$

$2\int_0^{\frac{1}{2}} dy + 4\int_{\frac{\sqrt{2}}{2}}^1 \frac{\sqrt{1-u^2}}{\cancel{u}}\cancel{u}\,du$

$4\int_{\frac{\sqrt{2}}{2}}^1 \sqrt{1-u^2}\,du$

Let $u = \sin\theta$

$du = \cos\theta\,d\theta$

$u = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$u = 1 \Rightarrow \theta = \frac{\pi}{2}$

$2Y\Big|_0^{1/2} + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{4}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + \cos 2\theta\,d\theta = 2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$

$1 + 2\left[\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\right] = 1 + \frac{\pi}{2} - 1 = \frac{\pi}{2}$

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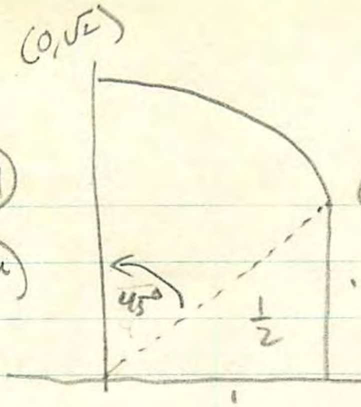
㉑ a)



(1,1) Area of Sector $= \frac{1}{2}\theta r^2 = \frac{1}{2}\frac{\pi}{4}\cdot 2 = \frac{\pi}{4}$

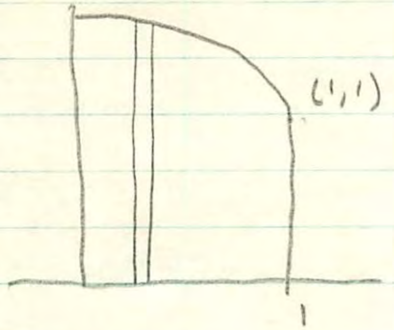
So area of figure is $\frac{1}{2} + \frac{\pi}{4}$

b)

Vertical

$x^2+y^2=2$

$y=\sqrt{2-x^2}$



$$\int_0^1 \sqrt{2-x^2}\,dx$$

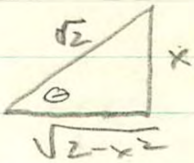
let $x^2 = 2\sin^2\theta$

$x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\,d\theta$

$x=0 \quad \theta = 0$

$x=1 \quad \theta = \frac{\pi}{4}$



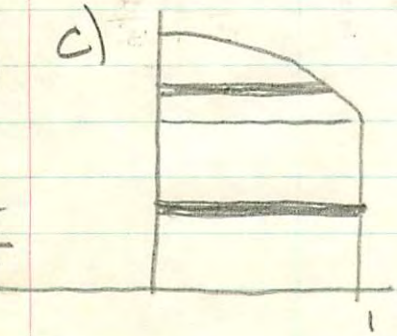
$$\sqrt{2}\cdot\sqrt{2}\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta = \frac{2}{2}\int_0^{\frac{\pi}{4}} 1+\cos 2\theta\,d\theta = \theta + \frac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}+\frac{1}{2}$$

c)

Horizontal

$x^2+y^2=2$

$x=\sqrt{2-y^2}$



$$\int_0^1 dy + \int_1^{\sqrt{2}} \sqrt{2-y^2}\,dy$$

$y = \sqrt{2}\sin\theta$

$y=1 \quad \theta = \frac{\pi}{4}$

$y=\sqrt{2} \quad \theta = \frac{\pi}{2}$

$$y\Big|_0^1 + \theta + \frac{1}{2}\sin 2\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$1 + \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} = \frac{1}{2}+\frac{\pi}{4}$$

NOTE: To integrate $\int\sqrt{2-y^2}\,dy$ or $\int\sqrt{2-x^2}\,dx$ as well as problem 19 we could have used formula #32

$$\int\sqrt{p^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{p^2-x^2}+p^2\sin^{-1}\frac{x}{p}\right)$$ thus: $\int_1^{\sqrt{2}}\sqrt{2-y^2}\,dy \Rightarrow$

$$\frac{1}{2}\left(y\sqrt{2-y^2}+2\sin^{-1}\frac{y}{\sqrt{2}}\right)\Big|_1^{\sqrt{2}} = \frac{1}{2}\left(\sqrt{2}(0)+2\sin^{-1}1.\right) - \frac{1}{2}\left(1+2\sin^{-1}\frac{1}{\sqrt{2}}\right) =$$

$$\frac{\pi}{2}-\frac{1}{2}-\frac{\pi}{4} = \frac{\pi}{4}-\frac{1}{2}$$

Now the complete area is:

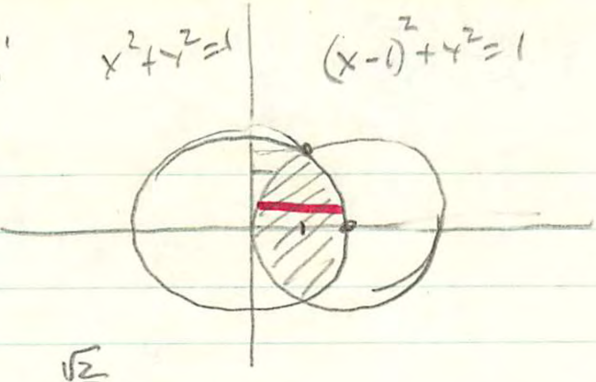
$$1+\frac{\pi}{2}-\frac{1}{2} = \frac{1}{2}+\frac{\pi}{2},$$

Pg 374

wrong!

㉓

This looks like a good pblm.



$$x^2+\cancel{y^2}=1=(x-1)^2+\cancel{y^2}$$

$$x=\pm(x-1)$$

$$\cancel{x^2}=\cancel{x^2}-2x+1$$

$$x=\frac{1}{2} \Rightarrow y=\pm\frac{\sqrt2}{2}$$

$$x=\sqrt{1-y^2}$$

$$(x-1)=\pm\sqrt{1-y^2}$$

$$x=1-\sqrt{1-y^2}.$$

$$2\int_0^{\frac{\sqrt2}{2}}\left[\sqrt{1-y^2}-\left(1-\sqrt{1-y^2}\right)\right]dy$$

$$2\int_0^{\frac{\sqrt2}{2}} 2\sqrt{1-y^2}-1\;dy$$

$$4\cdot\frac{1}{2}\left(y\sqrt{1-y^2}+\sin^{-1}y\right)\Big|_0^{\frac{\sqrt2}{2}}-2y\Big|_0^{\frac{\sqrt2}{2}}$$

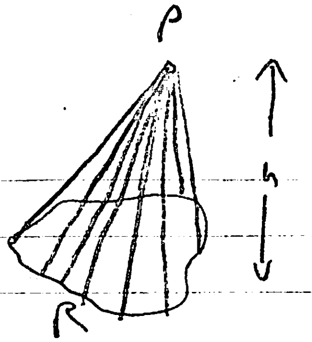
$$\cancel{2}\left(\sqrt2\,\frac{\sqrt2}{\cancel{2}}+\frac{\pi}{\underset{2}{\cancel{4}}}\right)-\sqrt2$$

$$2+\pi-\sqrt2$$

pg 374

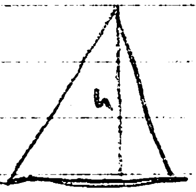
?

(30)



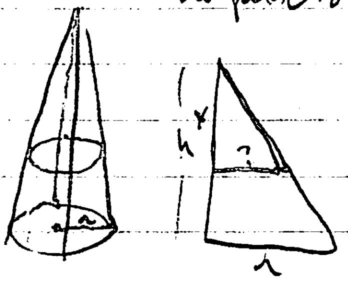
R has area A   WTS Volume $\frac{1}{3}Ah$

~~Typical cross section is a △ with height h.~~



consider horizontal cross sections. The bottom one is the base R. Set x axis vertically w/ 0 at base Then Peak is at h. as $x=0 \to x>h$ we have similar areas (but smaller) we would have similar figures, which have a ratio K so their areas have have a ratio of K^2

If this were a cone ! w radius r, height h then a typical horizontal cross section is a circle

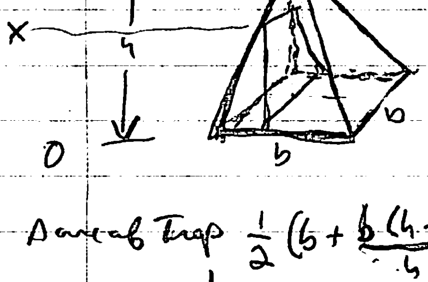


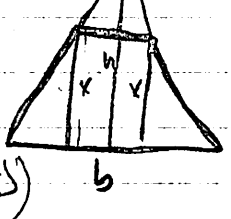
$? = \frac{xr}{h}$ so $\frac{\pi x^2 r^2}{h^2}$ is typical area

$$\int_0^h \frac{\pi x^2 r^2}{h^2}\,dx = \frac{1}{3}\frac{\pi r^2}{h^2}x^3\Big|_0^h = \frac{1}{3}\pi r^2 h$$

however base ~~A~~ has area πr^2 so this becomes $\frac{1}{3}Ah$.

If this were a Pyramid 58 base b×b typical cross section is is trapezoid Lower base b height x





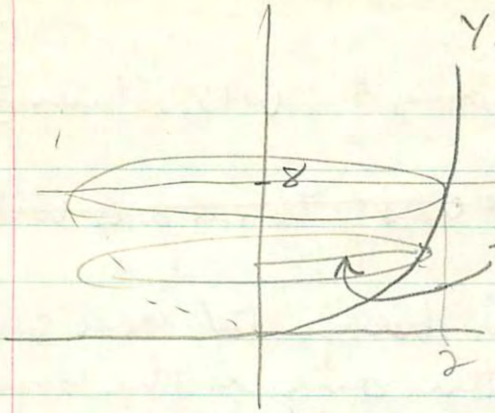
$\frac{h}{\frac{b}{2}} = \frac{h-x}{?} \Rightarrow \frac{2h}{b} = \frac{h-x}{?}$

$\frac{b}{2h} = \frac{?}{h-x}$ $? = \frac{b(h-x)}{2h}$

Area of Trap $\frac{1}{2}\left(b + \frac{b(h-x)}{h}\right)$

$$\cancel{\frac{1}{2}}2\int_0^h b + \frac{b(h-x)}{h}\,dx = \int_0^h 2b + -\frac{1}{h}\int x\,dx = 2bx\Big|_0^h - \frac{1}{2}\frac{x^2}{h}\Big|_0^h$$

$2bh - \frac{h}{2}$



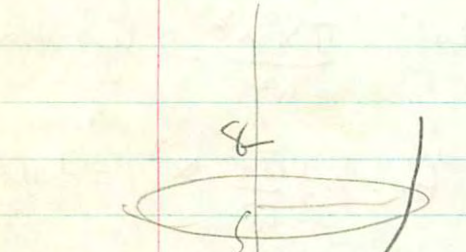
This radius is $\sqrt[3]{y}$

$$\boxed{\pi\int_0^8 y^{\frac{2}{3}}\,dy}$$

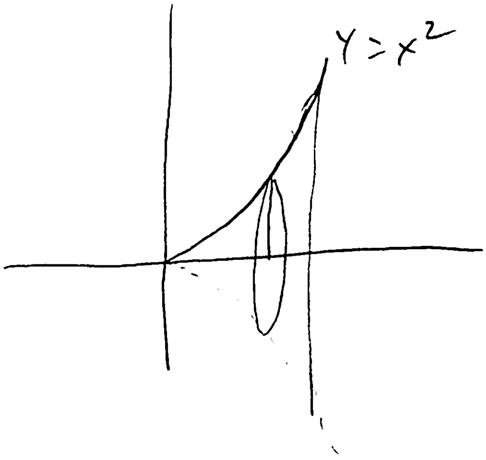
This is called "disks"

$$\pi\int_a^b r^2\,dx \rightarrow \cancel{\pi\int_0^2 x^6\,dx}$$

Those → are called "washers"



washers



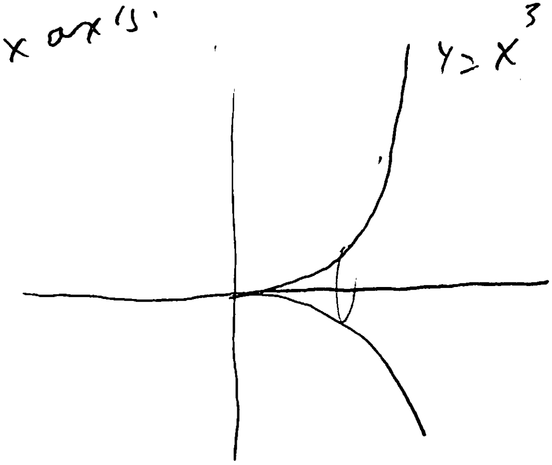
$\pi\int_0^1 r^2\,dx$

called "disks"

$\pi\int_0^1 x^2\,dx$

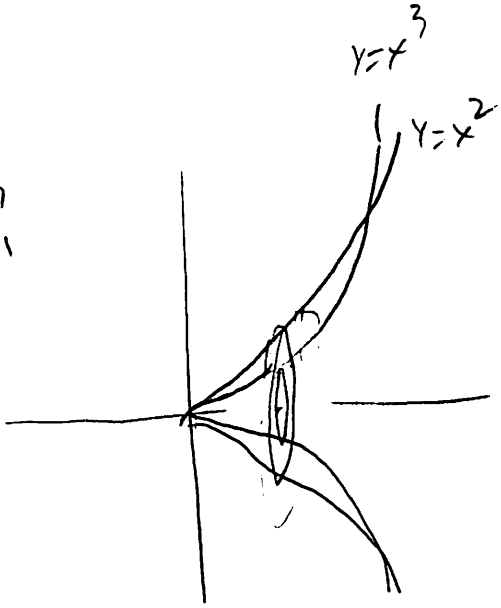
$\pi\frac{x^3}{3}\Big|_0^1 = \frac{\pi}{3}.$

Rotate around x axis.



$\pi\int_0^1 x^3\,dx$

$\pi\frac{x^4}{4}\Big|_0^1 = \frac{\pi}{4}$



which curve is which?

why?

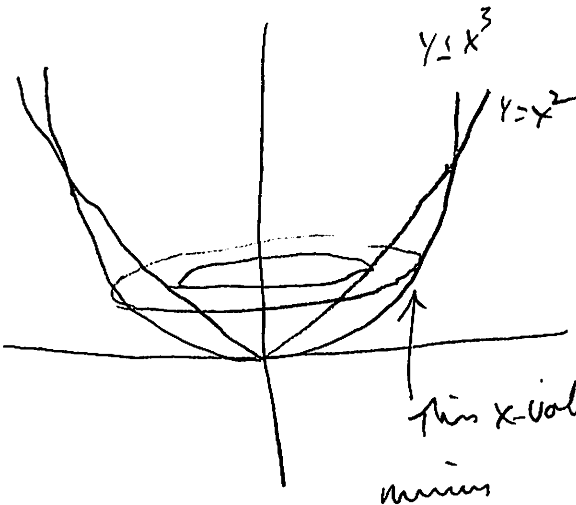
"washers"

$\pi\int_a^1 R^2 - r^2\,dx$

$\pi\int_0^1 x^2 - x^3\,dx$

also called "parallel cross sections"

"about y axis"



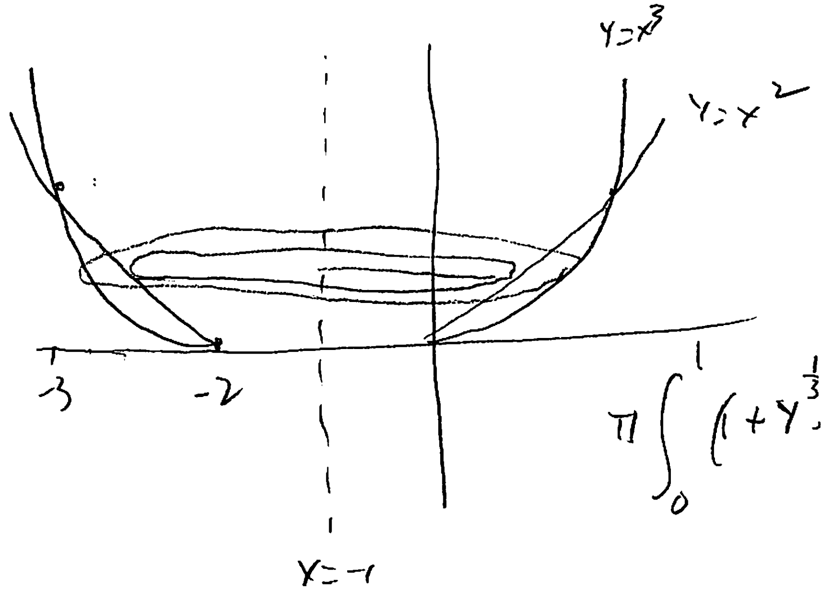
This x-value $\sqrt[3]{y}$

minus

This x-value is the radius.

$$\pi\int_{y=0}^{y=1} \left(y^{\frac{1}{3}}\right)^2 - \left(y^{\frac{1}{2}}\right)^2 dy$$

"about x = -1



$$\pi\int_0^1 \left(1+y^{\frac{1}{3}}\right)^2 - \left(1+y^{\frac{1}{2}}\right)^2 dy$$

$\frac{4\pi}{15}$

DISCS / Washers ~~aka Annular Rings~~

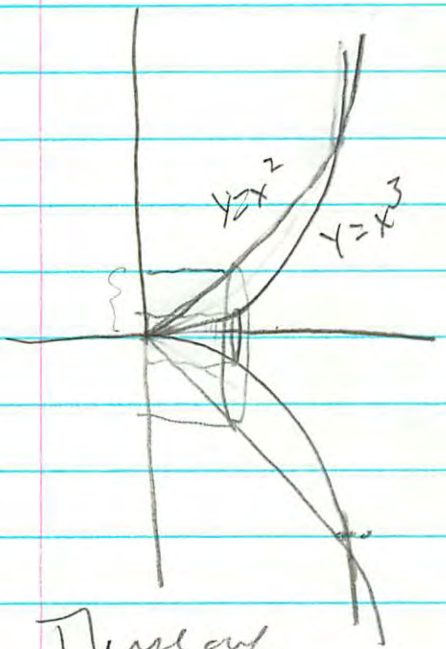
also called (Pg 380 pblm 17-19) parallel cross sections

Lecture washers

~~Before 7th~~ ~~Hw assignmt~~ Pg 374 21, 31-35

Consider region bounded by $y = x^2$ and $y = x^3$ for $0 \le x \le 1$

If this region is rotated



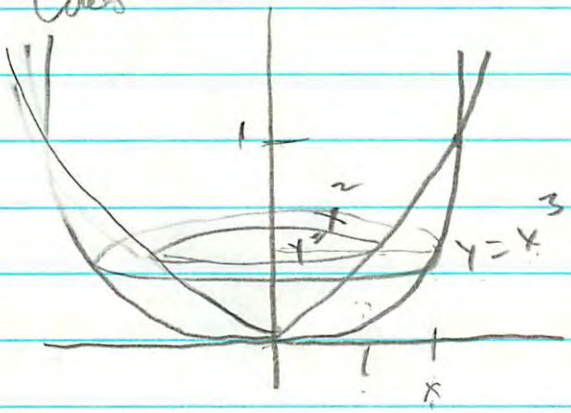
2 Typical cross sections circle radius x^2 − circle radius x^3

area $\pi(x^2)^2 - \pi(x^3)^2$

$$\pi\int_0^1 x^4 - x^6\,dx = \frac{\pi x^5}{5} - \frac{\pi x^7}{7}\Big|_0^1 =$$

$$\pi\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{2\pi}{35} = .180$$

These are called "washers".



$$\pi\int_0^1 \left(y^{\frac{1}{3}}\right)^2 - \left(y^{\frac{1}{2}}\right)^2 dy$$

$$\pi\int_0^1 y^{\frac{2}{3}} - y\,dy$$

$$\pi\left[\frac{3y^{\frac{5}{3}}}{5} - \frac{y^2}{2}\right|_0^1 =$$

$$\pi\left(\frac{3}{5} - \frac{1}{2}\right) = \pi\left(\frac{6-5}{10}\right)$$

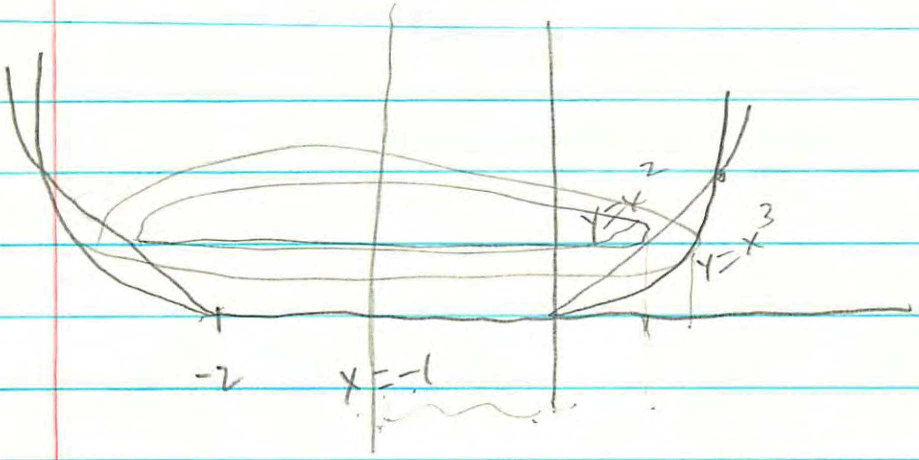
$$\frac{\pi}{10}$$

more ↓

~~GNDay~~ ~~washers~~

Suppose that regio is rotated about the vertical line $x=-1$

Now Show pblms 32, 33 pg 374



radius of rev are: $1+y^{\frac{1}{3}}$ and $1+y^{\frac{1}{2}}$

$$A(y)=\pi\left(1+y^{\frac{1}{3}}\right)^2-\pi\left(1+y^{\frac{1}{2}}\right)^2$$

$$=\pi\left[1+2y^{\frac{1}{3}}+y^{\frac{2}{3}}-1-2y^{\frac{1}{2}}-y\right]$$

$$=\pi\left[2y^{\frac{1}{3}}+y^{\frac{2}{3}}-2y^{\frac{1}{2}}-y\right]$$

$$\pi\int_0^1 2y^{\frac{1}{3}}+y^{\frac{2}{3}}-2y^{\frac{1}{2}}-y\;dy$$

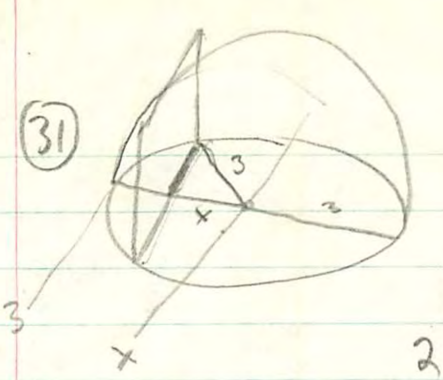
$$\pi\left[\frac{3}{2}y^{\frac{4}{3}}+\frac{3y^{\frac{5}{3}}}{5}-\frac{4y^{\frac{3}{2}}}{3}-\frac{1}{2}y^2\right]\Bigg|_0^1=$$

$$\pi\left[\frac{3}{2}+\frac{3}{5}-\frac{4}{3}-\frac{1}{2}\right]$$

$$\pi\,\frac{45+18-40-15}{30}\qquad\frac{8\pi}{30}\qquad\frac{4\pi}{15}$$

Pg 374

(31)



side of sq is $2\sqrt{9-x^2}$

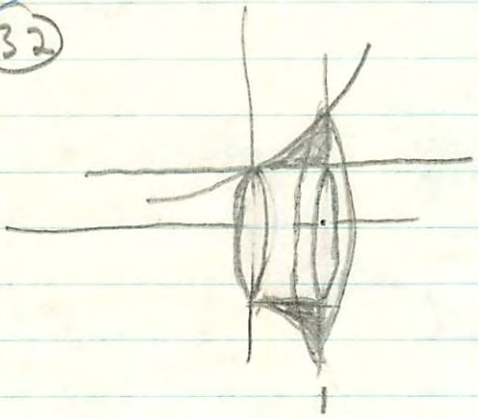
area of cross section is $4(9-x^2)$

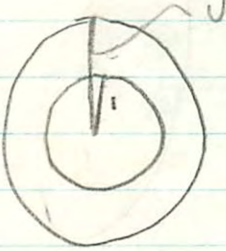
$$2\int_0^3 4(9-x^2)\,dx = 72x - \frac{8}{3}x^3\Big|_0^3 =$$

$$216 - 72 = 144.$$

washers

(32)





we need to subtract out cylinder $-\int_0^1 \pi\,dx = -\pi$
i.e. a circle fm $0 \to 1$

other term circle w/radius e^x as x goes fm $0 \to 1$

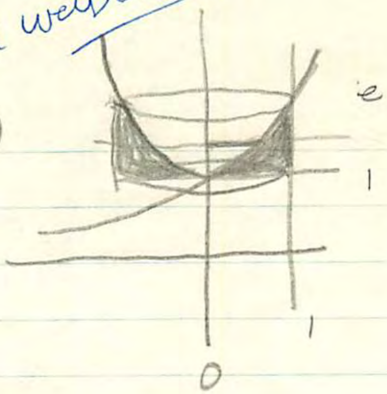
$$\int_0^1 \pi e^{2x} = \frac{\pi}{2}e^{2x}\Big|_0^1 = \frac{\pi}{2}e^2 - \frac{\pi}{2}$$

required volume is $\frac{\pi}{2}e^2 - \frac{\pi}{2} - \pi = \frac{\pi}{2}(e^2-3)$.

mere waylans

Pg 374

(33)



circle radius $e^x = y \;=\; \ln y = x$

$$-\int_1^e \pi(\ln y)^2 dy + \int_1^e \pi\, dy$$

$r = 1$
using crc #489

$$-\pi\left(y(\ln y)^2 - 2y\ln y + 2y\right)\Big|_1^e + \pi y\Big|_1^e$$

$$-\pi(e - \cancel{2e} + \cancel{2e} - 2) + \pi e - \pi$$

$$-\pi e + 2\pi + \pi e - \pi = \pi$$

w/out using tables: $\displaystyle\int (\ln y)^2 dy = \int \frac{y(\ln y)^2}{y} dy$

This method inspired by the need for $\frac{1}{y}dy$: "so we put it in" of course we are then obligated to remove it

let $p = \ln y$
$dp = \frac{1}{y} dy \Rightarrow \displaystyle\int e^p p^2 dp \Rightarrow$ let $u = p^2$ $\quad v = e^p$
$e^p = y$ $\qquad\qquad\qquad\qquad du = 2p\,dp \quad dv = e^p dp \Rightarrow$

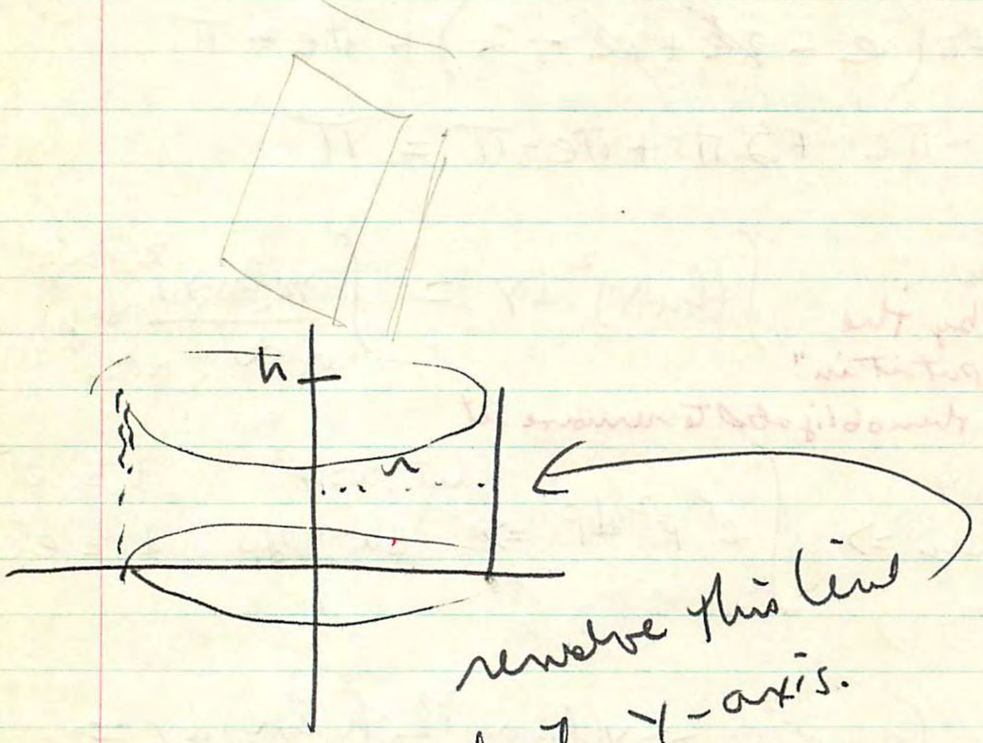
$$p^2 e^p - \int 2p e^p dp \Rightarrow y(\ln y)^2 - 2\int \frac{y(\ln y)}{y} dy =$$

$y(\ln y)^2 - 2\int \ln y\, dy$ $\quad$ let $u = \ln y$ $\quad v = y$
$\qquad\qquad du = \frac{1}{y}dy \quad dv = dy$

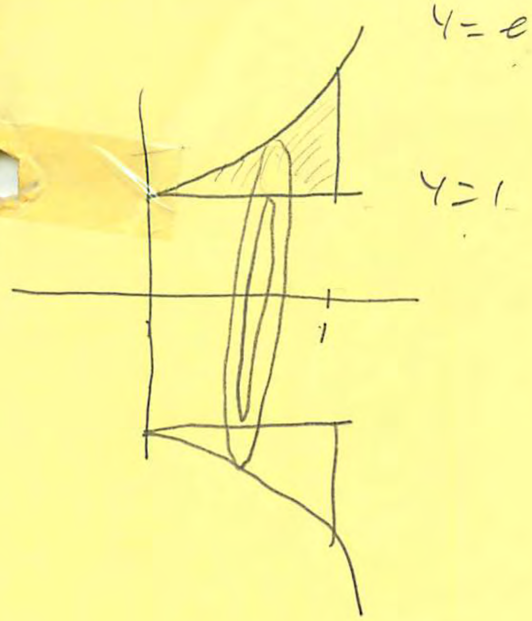
$$y(\ln y)^2 - 2\left[y\ln y - \int dy\right] = y(\ln y)^2 - 2y\ln y - 2y$$

Note: This could be done using parts w/out adding in the extra Y.

$u = \ln Y \quad dY$
$du = \frac{1}{Y} \quad dY$
$dv = Y$
$v = Y$ etc.



revolve this line
about Y-axis.
to get
Cylinder.
h is hieght of line
r is radii of arc
2π revolves it.



washers DISKS

Shell openers

$$\pi\int_0^1 e^{2x}-1\,dx$$

$$\pi\left(\frac{e^{2x}}{2}-x\right|_0^1$$

$$\pi\left(\frac{e}{2}-1-\frac{1}{2}\right)$$

$$\frac{\pi}{2}(e-3)$$

Review from Yesterday.

Shells opener. Pg 1

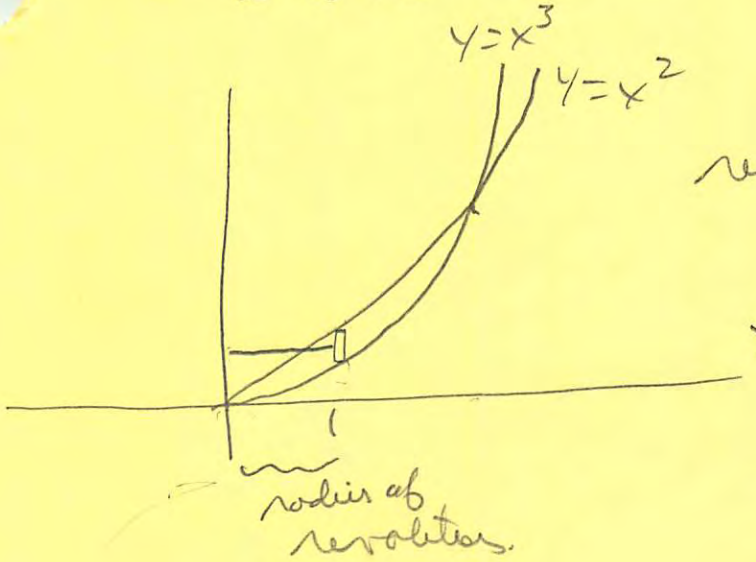
How do we find Surface area of cylinder?

$S = 2\pi r h$

why 2π? revolves a point r distance from center creating a circle.

what's h? length of typical cross section

what do you get if r is allowed to change from zero to radius of cylinder? volume?



revolve about y axis.

$$2\pi\int_0^1 x\,(x^2-x^3)\,dx$$

$$2\pi\int_0^1 x^3-x^4\,dx$$

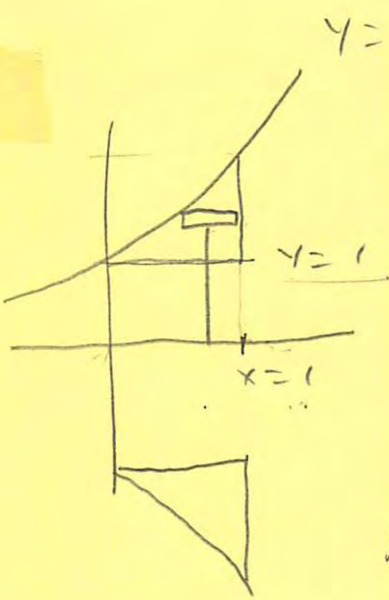
$$2\pi\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{20}\right) = \frac{\pi}{10}$$

Notice cross section is a rectangle.

Radius is a line.

Compare to Problem we did yesterday in notes.

Shells



cross section is a rectangle
radius of Rev is a line.

$$2\pi\int_1^e (1-\ln y)\,y\,dy$$

revolves

length of cross section

Radius of Rev

cross section moves from 1 to e up y axis.

$$2\pi\int_1^e y - y\ln y\,dy$$

$\int y\ln y$

$u = \ln y \quad dv = y\,dy$

$du = \frac{1}{y}\,dy \quad v = \frac{y^2}{2}$

$\frac{y^2}{2}\ln y - \frac{1}{2}\int y\,dy$

$$\downarrow$$

$$2\pi\left[\frac{y^2}{2} - \left(\frac{y^2}{2}\ln y - \frac{1}{4}y^2\right)\right]\Bigg|_1^e$$

$$2\pi\left[\frac{2y^2}{4} - \frac{y^2}{2}\ln y + \frac{y^2}{4}\right]\Bigg|_1^e$$

$$2\pi y^2\left(\frac{3}{4} - \frac{1}{2}\ln y\right)\Bigg|_1^e$$

$$2\pi\left[e^2\left(\frac{3}{4} - \frac{1}{2}\right) - \left(\frac{3}{4} - \frac{1}{2}\cdot 0\right)\right]$$

$$2\pi\left(\frac{e^2}{4} - \frac{3}{4}\right)$$

$$\frac{\pi}{2}\left(e^2 - 3\right)$$

more on Next Page.

Same prblm done in "X"

$2\pi\int_1^e y - y\ln y\, dy$

$y = e^x$

$dy = e^x\,dx$

$y = 1 \quad x = 0$

$y = e \quad x = 1$

$2\pi\int_1^e \underset{\downarrow}{y}(1-\underset{\downarrow}{\ln y})\,\underset{\downarrow}{dy}$

$2\pi\int_0^1 e^x(1-x)\,e^x dx$

$2\pi\int_0^1 e^{2x}(1-x)\,dx$

$u = 1-x \qquad dv = e^{2x}dx$

$du = -dx \qquad v = \frac{1}{2}e^{2x}$

$2\pi\left[\frac{1}{2}(1-x)e^{2x}\Big|_0^1 + \frac{1}{2}\int_0^1 e^{2x}dx\right]$

$2\pi\left[\frac{1}{2}(1-x)e^{2x} + \frac{1}{4}e^{2x}\right]\Big|_0^1$

$2\pi\left[\frac{1}{2}(0)e^2 + \frac{1}{4}e^2 - \left(\underset{\frac{1}{2}+\frac{1}{4}}{\frac{1}{2}(1)(1)+\frac{1}{4}\cdot 1}\right)\right]$

$2\pi\left[\frac{1}{4}e^2 - \frac{3}{4}\right]$

$\frac{\pi e^2}{2} - \frac{3\pi}{2}$

$\frac{\pi}{2}(e^2-3)$

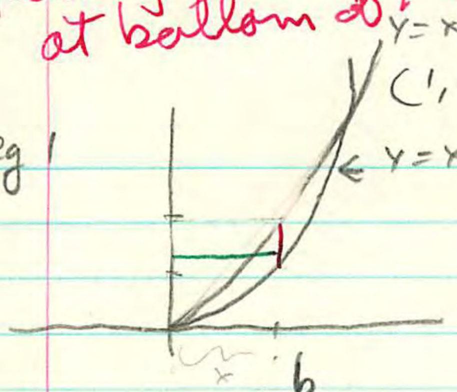
End of lecture

Shells technique

Pg 375-379

Opening at bottom of page.

eg 1



the red is the cross-section. the green is the radius of revolution.

$$2\pi\int_a^b (\text{radius of revolution})(\text{length of cross section})\,dx$$

revolves The cross section

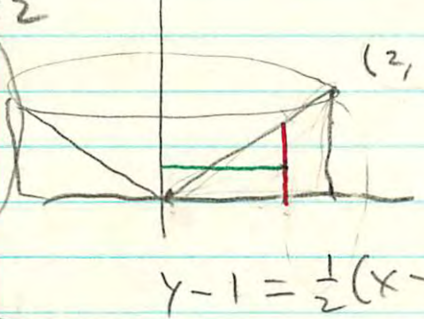
Looking from the top we see each of the "shells" or cylinders as rings of a tree stump. we add up all the surface areas of these cylinders to get the Volume of the item.

Thus

$$2\pi\int_0^1 x\,(x^2-x^3)\,dx$$

eg 2

Contrast with washers



$$2\pi\int_0^2 x\cdot\tfrac{1}{2}x\,dx$$

$y-1=\tfrac{1}{2}(x-2)$

$y=\tfrac{1}{2}x$

$2y=x$

$$\pi\int_0^2 x^2\,dx = \pi\frac{x^3}{3}\Big|_0^2 = \frac{8\pi}{3}$$

OPENING:

Open with "how do you find surface area of cylinder?

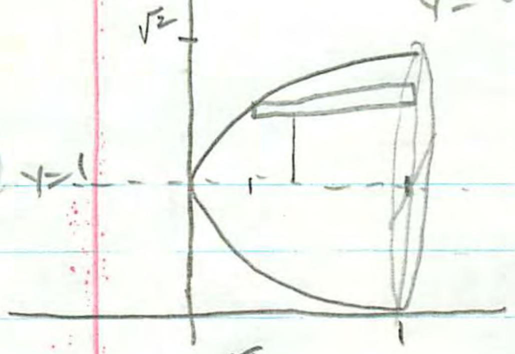
$S = 2\pi r h$ why 2π? revolve a pt around it make circumference.

RADIUS OF Revolution

length of "TYPICAL" cross section.

If R changes from 0 to radius of cylinder what do we get? Volume.

Pg 380
#12



$y = \sqrt{1+x^2}$

$y^2 = 1 + x^2$

$x^2 = y^2 - 1$

$x = \sqrt{y^2-1}$

$$2\pi \int_1^{\sqrt{2}} (y-1)\left(1-\sqrt{y^2-1}\right) dy$$

$$2\pi \left[\int_1^{\sqrt{2}} y-1 \, dy + \int_1^{\sqrt{2}} \sqrt{y^2-1} \, dy - \int_1^{\sqrt{2}} y\sqrt{y^2-1} \, dy \right]$$

☆

$$2\pi \left[\frac{y^2}{2} - y \Big|_1^{\sqrt{2}} + \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\frac{\pi}{4}} - \frac{u^{\frac{3}{2}}}{3}\Big|_0^1 \right]$$

$$2\pi \left[1 - \sqrt{2} - \left(\frac{1}{2} - 1\right) + \frac{1}{2}\left(\sqrt{2}\cdot 1 - \ln(\sqrt{2}+1) - 1\cdot 0 - \ln(1+0)\right) - \frac{1}{3} + 0 \right]$$

$$2\pi \left[1 - \sqrt{2} + \frac{1}{2} + \frac{\sqrt{2}}{2} - \frac{1}{2}\ln(\sqrt{2}+1) - 0 - \frac{1}{3} \right]$$

$$2\pi \left[\frac{3}{2} - \frac{1}{3} - \frac{\sqrt{2}}{2} - \frac{1}{2}\ln(\sqrt{2}+1) \right]$$

$$2\pi \left[\frac{9-2}{6} - \text{etc} \right.$$

$$2\pi \left[\frac{7}{6} - \text{etc} \right.$$

$$\pi \left[\frac{7}{3} - \sqrt{2} - \ln(\sqrt{2}+1) \right]$$

☆

$$\int_1^{\sqrt{2}} y\sqrt{y^2-1} \, dy \rightarrow \frac{1}{2}\int_0^1 u^{\frac{1}{2}} du$$

$$\cancel{\frac{1}{2}}\left(\frac{\cancel{2}u^{\frac{3}{2}}}{3}\right)$$

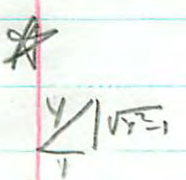
$y^2 - 1 = u$

$2y\,dy = du$

$y\,dy = \frac{1}{2} du$

$y = 1 \quad u = 0$

$y = \sqrt{2} \quad u = 1$



$y = \sec\theta$

$dy = \sec\theta\tan\theta$

$y = 1 \quad \theta = 0$

$y = \sqrt{2} \quad \theta = \frac{\pi}{4}$

$s^2 + c^2 = 1$

$t^2 = \sec^2 - 1$

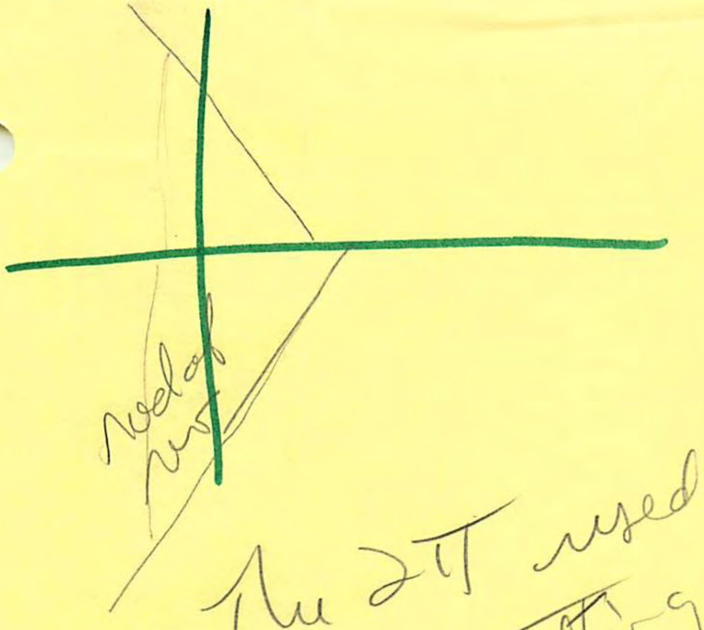
$$\int \tan\theta(\sec\theta\tan\theta)\,d\theta$$

$$\int \sec\theta\tan^2\theta \, d\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta - \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) - \ln|\sec\theta + \tan\theta|$$

$$\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| \Big|_0^{\frac{\pi}{4}}$$



The 2π used in "rotating a line about some axis" refers to a line "above x-axis".

Average of fctn over an interval

10.5
pg 380 – 384

following the discussion in the text pg 380, we define the average of a fctn over $[a, b]$, That is, the average value of f over $[a, b]$ as

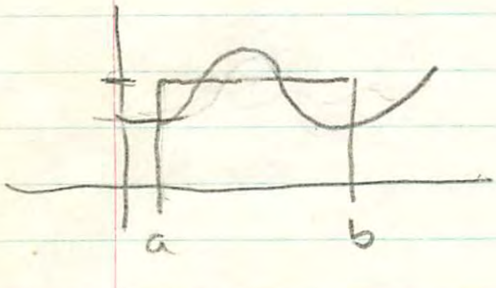
$$\frac{\int_a^b f(x)\,dx}{b-A}$$

we are convinced that the use of the integral is proper Because reasoning by the similarity to our development of the integral in chapt 7.

If A is this average value of fctn then

$$A = \frac{\int_a^b f(x)\,dx}{b-a} = \underset{\substack{\uparrow \\ \text{must be altitude.}}}{A}\underbrace{(b-a)}_{\text{base}} = \underbrace{\int_a^b f(x)\,dx}_{\text{area under curve}}$$

Now looking at this graphically: A is the height of the rectangle. The rectangle has the same area as that area under the curve.



eg 1 average of $\sin x$ for $x \in [0, \pi]$

$$A = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(-1) + \frac{1}{\pi}(1) = \frac{2}{\pi}$$

$\frac{2}{\pi} \doteq .64$ which may be surprising since f goes from 0 to 1 and back to zero. .5 would have been my guess. but sin curve Not semi circle.

10.5
Pg 380-384

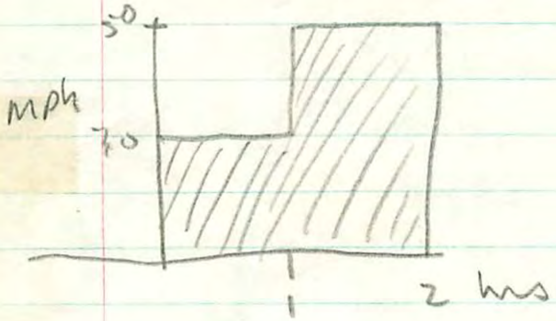
eg 2 If we travel 30 mph for 1 hr then 50 mph for 1 hr, what is our average velocity wrt time?

Arithmetic: Average = $\frac{\text{total dist}}{\text{total time}} \Rightarrow$

$$\frac{30\cdot 1 + 50\cdot 1}{1+1} = \frac{80}{2} = 40 \text{ mph.}$$

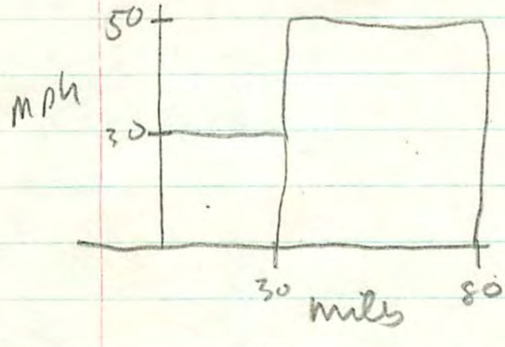
Calculus: $[0,2]$, $\int_0^2 v(t)\,dt$. in order to evaluate

This integral (we do not have the fctn v) we can think of it as the area under the curve so



$$\frac{\int_0^2 v(t)\,dt}{b-a} = \frac{30+50}{2-0} = \frac{80}{2} = 40$$

consider however: Same Pblm. we travel 30 mph for 30 miles then we travel 50 mph for 50 miles.



Now $$\frac{\int_0^{80} g(x)\,dx}{80-0} = \frac{\overset{15}{\cancel{30}}\cdot 3\cancel{0} + \overset{25}{\cancel{50}}\cdot 5\cancel{0}}{\underset{4}{\cancel{80}}} =$$

$$\frac{45+125}{4} = \frac{170}{4} = 42.5 \text{ mph.}$$

Why do these averages differ? It can be shown that the average wrt time is always less than or equal to the average wrt distance. (exercise 12, 13) The chief cause for the difference is that more distance is traveled and this "weighting" is

10.5
Pg 380-384

reflected in the increased average value. "Since greater distances are travelled in equal time when v is large, the averaging WRT distance space gives a heavier weighting to the higher velocities. When an average velocity is specified without further mention of the method of averaging, then the average WRT time should be understood."

See June 1979 2 YR College Journal.

Consider How to evaluate voting Power in a weighted voting body.

Let: $\{q; w_1, w_2, w_3, \ldots w_n]$ represent a weighted voting body where there are n voters, the ith voter casts w_i votes and q votes are needed to pass a bill.

eg 1 $\{7; 4, 3, 2, 1\}$ represents a body, call them A, B, C, D casting 4, 3, 2, 1 votes respectively and 7 votes are needed to Pass a bill.

This is common in such things as Stock holders votes, United Nations, World Bank etc. (New Mexico legislature is another)

In our example voter A has 40% of the vote. B Has 30%, C 20% and D 10%. How much Power Does A have? 40%? DID You Notice that he has Veto Power? How much is that worth?

eg 2 $\{6; 7, 1, 1, 4\}$ What could You say about A? Dictator?

Power is

f(P) is the ~~likelihood~~ probability a member's vote will make a difference.

NEXT

we define a fctn f(P)

where f is the probability that a member's vote will make a difference in the vote.

Finally we define

~~Then~~ Power ~~will be defined to be~~

as the probability a member's vote will make a difference

→ Averaged over all probabilities.

(ie from 0 to 1)

"averaged over all votes" would not be a weighted average.

Astrone
Pg 380-384

eg 3 $\{6; 3, 3, 3, 1\}$: what could you say about D? Dummy? Clearly D has absolutely No effect on the outcome of any vote.

It should be reasonable that the measure of Voting Power is more than proportions of Votes.

Let us suppose that each Bill which comes before the Voting body can be assigned a Number "p" $\in [0,1]$, where p is the probability that any member will vote for the Bill. Think of "p" as the level of acceptability. ie Some bills will be unacceptable, "p" Near 0, and will usually (but not always) fail, others will be generally acceptable, "p" Near 1 and will usually pass while others will be very controversial and "p" Near $\frac{1}{2}$.

Now let us suppose we want to know how much power a Voter has. This power will in most cases vary as a fctn of P. 1ST we Need to Define Power so the idea of Power (as a fctn of P) will be reasonable.

Power is the probability that a certain member's vote will make a difference to the outcome on a bill with acceptability p, averaged over all p between 0 and 1.

Cetnar
Pg 380 - 384

The idea of integration comes in when we take the Average over $[0, 1]$.

eg 4 $\underset{\quad A \quad B \quad C}{[3; 2, 1, 1]}$

Suppose a bill with acceptability p comes before the body. Thus each member votes for the bill with Probability p.

Notice A's vote is crucial however, if B and C Vote No it makes no difference what A votes.

So A's vote will make a difference;

$$f_A(p) = \underset{B\text{ yes } C\text{ No}}{p \overset{\text{AND}}{\,} (1-p)} \overset{\text{OR}}{+} \underset{B\text{ No } C\text{ Yes}}{(1-p) \overset{\text{And}}{\,} (p)} \overset{\text{or}}{+} \underset{B\text{ yes } C\text{ Yes}}{p \overset{\text{And}}{\cdot} p}$$

$$= p - p^2 + p - p^2 + p^2 = 2p - p^2$$

B's vote will make a difference only if A Votes Yes and C Votes No so

$$f_B(p) = \underset{A\text{ yes } C\text{ No}}{p \overset{\text{and}}{\,} (1-p)} = p - p^2$$

also we have the same for $f_C(p)$

So $f_C(p) = p - p^2$

f is the probability that its member's vote will make a difference in the outcome of a particular bill which has acceptability p.

all Y   B Yes c No d No + B Yes c Yes d No + B Yes c No d Y

$p^3 + \quad p(1-p)^2 \quad + \quad p^2(1-p) \quad + p^2(1-p)$

B No c Y D Y

$(1-p)p^2$

$$f_A(p) = 3p^2(1-p) + p(1-p)^2$$

$$p^3 + 3p^2 - 3p^3 + p - 2p^2 + p^3$$

$$\int_0^1 -p^3 + p^2 + p \, dp$$

$$\frac{-p^4}{4} + \frac{p^3}{3} + \frac{p^2}{2} \Big|_0^1$$

$$-\frac{1}{4} + \frac{1}{3} + \frac{1}{2} =$$

$$\underline{-3 + 4 + 6} = \frac{7}{12} = 58\%$$

Calculus Ch 10

How much power is *your* vote worth?

(aka .... Who put the DUH in Florida?)

J. Mumaugh

a couple of definitions ...

We define $f_i(p)$ as the probability that voter i will vote for a particular bill which has "acceptability" p.

We define "voting power" as the probability that voter i's vote will make a difference on a single bill *averaged for all possible probabilties.*

an example ...

The situation: It takes 3 votes to pass the bill. Voter "A" has 2 votes. Voters "B" and "C" each have one vote. This is summarized as: {3;2,1,1}

How much power do voters A, B, and C have?

The probability that Voter A's vote makes a difference is if Voter B votes Yes and Voter C votes No or if Voter A votes No and Voter B votes Yes or If Voter B votes Yes and Voter C votes Yes. This is summarized as:

$$\frac{1}{1-0}\int_0^1 2p-p^2dp = p^2-\frac{1}{3}p^3\Big|_0^1 = \frac{2}{3} \quad f_A(p)=p(1-p)+(1-p)p+p\cdot p$$

Notice that if Voter B votes no and Voter C votes no, then Voter A's vote has no meaning and is therefore not included in this list.

Now we integrate for all possibilities ... $\frac{1}{1-0}\int_0^1 2p-p^2dp = p^2-\frac{1}{3}p^3\Big|_0^1 = \frac{2}{3}$

Conclusion: Voter A has 66.7% of the power in this situation while Voter A has 50% of the vote! Does that make Voter A a dictator? a king? Does Voter A have veto power? ....

0 **Given the following voting situations ....**

#1 { 6 ; 7 , 1 , 1 , 1 }

#2 { 6 ; 3 , 3 , 3 , 1 }

#3 { 7 ; 4 , 3 , 2 , 1 }

#4 { 6 ; 4 , 3 , 2 , 1 }

#5 { 5 ; 4 , 3 , 2 , 1 }

#6 { 4 ; 4 , 3 , 2 , 1 }

#7 { 3 ; 4 , 3 , 2 , 1 }

#8 { 2 ; 4 , 3 , 2 , 1 }

Describe the power for each voter in each situation. Include all computations and explanations in support of your conclusions.

f is the probability that its member's vote will make a difference on a single bill with acceptability p.

P_g 380-384 Lecture and Hw about f I [illegible] Faye

Power is traditionally denoted by Φ (Phi) thus Define Power to be, the average value of f over all probabilities.

$$\Phi_A = \frac{\int_0^1 f_A(p)}{1-0} = \frac{\int_0^1 (2p-p^2)\,dp}{1-0} = p^2 - \tfrac{1}{3}p^3 \Big|_0^1 = \tfrac{2}{3}$$

$$\Phi_B = \Phi_C = \int_0^1 (p-p^2)\,dp = \tfrac{1}{2}p^2 - \tfrac{1}{3}p^3 \Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

So A has $\frac{2}{3}$ of the Power while B and C have $\frac{1}{6}$ each. this is reasonable since A has veto Power. A however is Not a Dictator since A cannot pass ANY thing he wants,

Students are to determine the Power for egs 1, 2, 3 above,

① [7; 4, 3, 2, 1]

A if B Yes or B No and C and D Yes

$$f_A(p) = p + (p-1)p \cdot p = p + p^3 - p^2$$

⟵ $$\Phi_A = \int_0^1 (p - p^2 + p^3)\,dp = \tfrac{1}{2}p^2 - \tfrac{1}{3}p^3 + \tfrac{1}{4}p^4 \Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} = \frac{6-4+3}{12} = \frac{7}{12} = 58\%$$

B's make a diff if A Yes C, D No or A Yes C No D Yes or A Yes C Yes D No

$$f_B(p) = p(1-p)(1-p) + p(1-p)p + p(p)(1-p)$$
$$= p - 2p^2 + p^3 + p^2 - p^3 + p^2 - p^3 = p - p^3$$
$$\Phi_B = \int_0^1 (p - p^3)\,dp = \tfrac{1}{2}p^2 - \tfrac{1}{4}p^4 \Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4} = 25\%$$

C's make a diff if A Yes B No D Yes $f_C(p) = p(1-p)p = p^2 - p^3$

$$\Phi_C = \int_0^1 p^2 - p^3\,dp = \tfrac{1}{3}p^3 - \tfrac{1}{4}p^4 \Big|_0^1 = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12} = 8\%$$

D's makes a diff if A Yes B No C Yes is Same as C ∴ 8%

↓ AB CD

#1 {6; 7,1,1,1}

#2 {6; 3,3,3,1}

#3 {7; 4,3,2,1}

#4 {6; 4,3,2,1}

#5 {5; 4,3,2,1}

#6 {4; 4,3,2,1}

#7 {3; 4 3 2 1}

#8 {2; 4 3 2 1}

#	A	B	C	D
1	100	0	0	0
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
3	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$
4	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$
5	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$
6	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$
7	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
8	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

write "Justifications" for your results.

in #1, A is a dictator he decides everything!

#3; D student rep to school Board.

#3, #6 have same solution. in #3 A has veto power in 6 A gets what he wants but cannot stop a coalition of others.

Pg 380-384 HW about Lecture

eg 2 $\{6; 7, 1, 1, 1\}$ A's only vote that counts ie makes no diff ∴ $\Phi_A = 1$

$\Phi_B = \Phi_C = \Phi_D = 0$

eg 3 $\{6; 3, 3, 3, 1\}$

A's makes diffence if B Yes C No or B No C Yes

$f_B(p) = p(1-p) + (1-p)(p) = 2p - 2p^2$

$\Phi_A = \int_0^1 2p - 2p^2 \, dp = p^2 - \frac{2}{3}p^3 \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

B is Same as A as is C ie

$A = \frac{1}{3}$ $B = \frac{1}{3}$ $C = \frac{1}{3}$ $D = 0$

Extra $\{6; 4, 3, 2, 1\}$ (A B C D)

$f_A(p) = p(1-p)^2 + p^2(1-p) + p(1-p)^2$

B Yes C No D No; B No C Yes D Yes; B No C Yes D No

$p^2(1-p)$ B Yes C No D Yes

$p^2(1-p)$ B Yes C Yes D No

$2p(1-p)^2 + 3p^2(1-p)$

$2p(1-2p+p^2) + 3p^2 - 3p^3$

$2p - 4p^2 + 2p^3 + 3p^2 - 3p^3$

$\int 2p - p^2 - p^3 \, dp$

$p^2 - \frac{p^3}{3} - \frac{p^4}{4} \Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{4}$

41.7% $\frac{12-7}{12} = \frac{5}{12}$

$f_B(p) = p(1-p)^2 + (1-p)p^2$

A = Yes C = No D = No; A = No C = Yes D = Yes

$p - 2p^2 + p^3 + p^2 - p^3$

$\int p - p^2 \, dp = \frac{p^2}{2} - \frac{p^3}{3} \Big|_0^1$ $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ $16\frac{2}{3}\%$

$f_C(p) = p^2(1-p)$ A = Yes B = No D = Yes; $p(1-p)^2$ A Yes B No D No; $p^2(1-p)$ A No B Yes D Yes

$2p^2 - 2p^3 + p - 2p^2 + p^3$

$\int p - p^3 \, dp = \frac{p^2}{2} - \frac{p^4}{4}$

$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ 25%

$f_D(p) =$ $p^2(1-p)$ A No B Yes C Yes

$\int p^2 - p^3 \, dp$

$\frac{p^3}{3} - \frac{p^4}{4} \Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}\%$

$8\frac{1}{3}\%$

#7 $\{3; 4, 3, 2, 1\}$

f_A

B NO	B NO	B NO
C YES	C NO	C NO
D NO	D YES	D NO

$(1-p)^2 p + (1-p)^2 p + (1-p)^3$

$(1-p)^2 (p + p + 1 - p)$

$(1-p)^2 (1+p)$

$(1-p^2)(1-p)$

$1 - p^2$
$-p + p^3$

$\int_0^1 p^3 - p^2 - p + 1\,dp$

$\frac{p^4}{4} - \frac{p^3}{3} - \frac{p^2}{2} + p \Big|_0^1$

$\frac{3 - 4 - 6 + 12}{12}$

$\frac{5}{12}$

f_B Same

f_C

A NO
B NO
D YES

$(1-p)^2 p = p - 2p^2 + p^3$

$\int_0^1 p^3 - 2p^2 + p\,dp$

$\frac{p^4}{4} - \frac{2p^3}{3} + \frac{p^2}{2} \Big|_0^1$

$\frac{3 - 8 + 6}{12} = \frac{1}{12}$

f_D Same as f_C

#8

$\{2;\ 4, 3, 2, 1\}$

f_A	B NO	B NO
	C NO	C NO
	D NO	D YES

$(1-p)^3 + (1-p)^2 p$

$(1-p)^2(1-p+p)$

$(1-p)^2$

$$\int_0^1 p^2-2p+1 \, dp$$

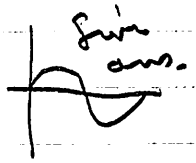
$$\frac{p^3}{3} - p^2 + p \Big|_0^1$$

$$\frac{1}{3} - 1 + 1 = \frac{1}{3}$$

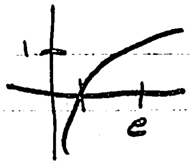
Pg 383

① x^2; $[-1, 3]$ $\frac{1}{3+1}\int_{-1}^{3} x^2\,dx = \frac{1}{4}\cdot\frac{1}{3}x^3\Big|_{-1}^{3} =$

$\frac{9}{4} + \frac{1}{12} = \frac{28}{12} = \frac{7}{3}$


give ans.

② $\sin x$ $[0, 2\pi]$ $\frac{1}{2\pi}\int_{0}^{2\pi}\sin x\,dx = \frac{-1}{2\pi}\left(\cos x\Big|_{0}^{2\pi}\right) = 0$



③ $\ln x$, $[1, e]$ $\frac{1}{e-1}\int_{1}^{e}\ln x\,dx = \frac{1}{e-1}\left(x\ln x - x\Big|_{1}^{e}\right)$

$\frac{1}{e-1}\left[(e-e)-(0-1)\right] = \frac{1}{e-1} \approx .582$

give ans

④ $\frac{1}{1+x^2}$ $[0, 1]$ $\int_{0}^{1}\frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_{0}^{1} = \frac{\pi}{4}$

⑤ $\frac{1}{1+x}$ $[1, 2]$ $\int_{1}^{2}\frac{dx}{1+x} = \ln|1+x|\Big|_{1}^{2} = \ln 3 - \ln 2 =$
$\ln\frac{3}{2}$

give ANS

⑥ $\sin^2 x$, $[0, \frac{\pi}{2}]$ $\frac{1}{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\sin^2 x\,dx = \frac{\cancel{2}}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{1}{\cancel{2}}(1-\cos 2x)\,dx =$

$\frac{1}{\pi}\left[\left(x - \frac{1}{2}\sin 2x\right)\Big|_{0}^{\frac{\pi}{2}}\right] = \frac{1}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{2} - \frac{1}{2\pi}\cdot 0 - 0 + 0\right] = \frac{1}{2}$

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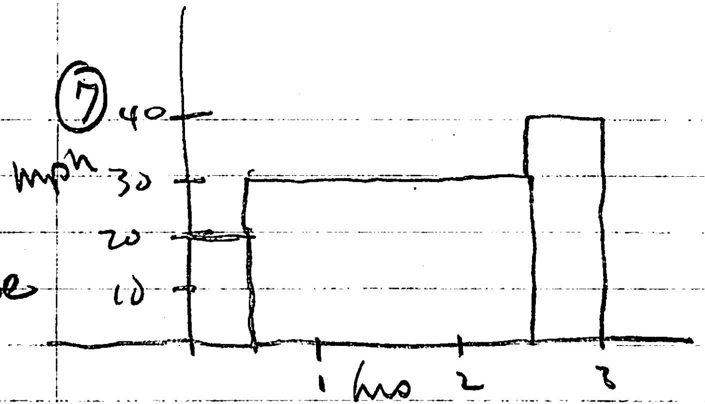
⑦

WRT Time



IN Both methods

$$\int \text{velocity}\, d(\;)$$

ie, WRT determines how we Look at limits of $\int$

$$\frac{\int_0^{\frac{1}{2}} 20\,dt + \int_{\frac{1}{2}}^{\frac{5}{2}} 30\,dt + \int_{\frac{5}{2}}^{3} 40\,dt}{3-0} = \frac{20t\Big|_0^{\frac{1}{2}} + 30t\Big|_{\frac{1}{2}}^{\frac{5}{2}} + 40t\Big|_{\frac{5}{2}}^{3}}{3} =$$

$$\frac{10 + 75 - 15 + 120 - 100}{3} = \frac{90}{3} = 30 \text{ mph}$$

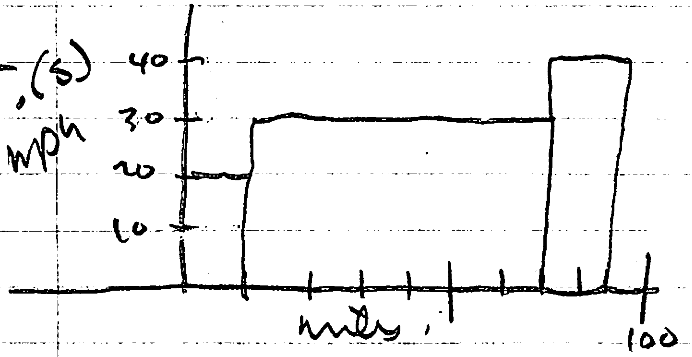
Velocity.

20 mph $\frac{1}{2}$ hr = 10 miles

30 mph 2 hrs = 60 miles.

40 mph $\frac{1}{2}$ hr = 20 miles

WRT dist, (s)



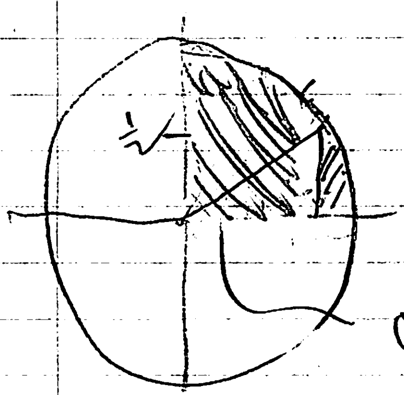
$$\frac{\int_0^{10} 20\,ds + \int_{10}^{70} 30\,ds + \int_{70}^{90} 40\,ds}{90-0} = \frac{20s\Big|_0^{10} + 30s\Big|_{10}^{70} + 40s\Big|_{70}^{90}}{90} =$$

$$\frac{1}{90}\left[200 + 2100 - 300 + 3600 - 2800\right] = \frac{280\cancel{0}}{9\cancel{0}} = 31.11$$

$$\frac{270}{9} + \frac{10}{9} = 31\tfrac{1}{9}.$$

⑨ⓒ

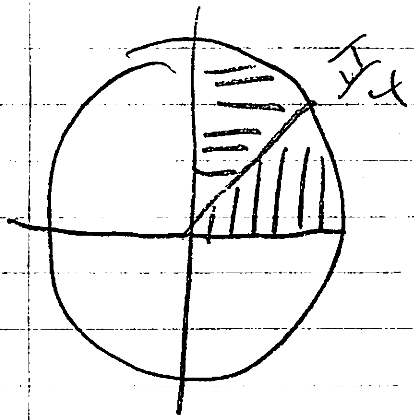
A)



corresponding to $\theta = 30^0$

$\pi/6$

B)



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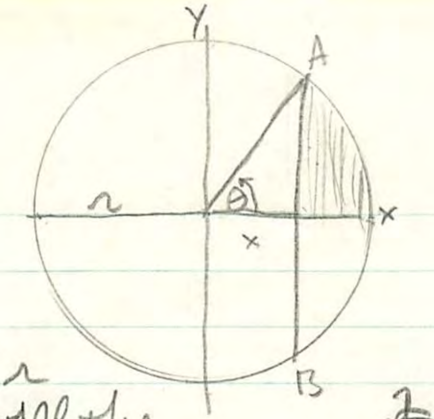
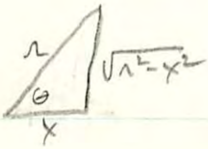
⑨



a)



$$AB = \frac{2\int_0^r \sqrt{r^2-x^2}\,dx}{r-0}$$

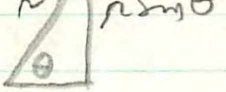
x goes fm 0 to r
and this covers ~~all~~ all the
lenghts of AB,

$$\frac{2}{r}\int_0^r \sqrt{r^2-x^2}\,dx =$$

$$\frac{\cancel{2}}{r}\cdot\frac{1}{\cancel{2}}\left(x\sqrt{r^2-x^2}+r^2\sin^{-1}\frac{x}{r}\right)\Big|_{-r}^{r} = \frac{1}{r}\left(r^2\cdot\frac{\pi}{4}-r^2\left(-\frac{\pi}{4}\right)\right)=$$

$$\frac{1}{\cancel{2}r}\left(\frac{\cancel{2}r^2\pi}{4}\right)=\frac{r\pi}{2} \qquad 1,570r$$

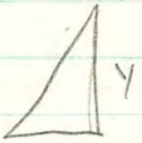
b)



$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} 2r\sin\theta\,d\theta = -\frac{4r}{\pi}\cos\theta\Big|_0^{\frac{\pi}{2}} = 0-\left(-\frac{4r}{\pi}\right)=\frac{4r}{\pi}$$

$$1,273R \qquad \frac{4r}{\pi}$$

c)



$$\frac{2\int_0^r Y\,dy}{r-0} = \frac{Y^2\Big|_0^r}{r} = \frac{r^2}{r}-0 = r,$$

This one is "what you expect the average length" to be,

D) in part a the area over $0<x<\frac{r}{2}$ is much larger than the area ~~over~~ $\frac{r}{2}<x<r$, in Part b the area over $\overset{0<\theta<\frac{\pi}{4}}{0<x<\frac{r}{2}}$ is equal to the area $\overset{\frac{\pi}{4}<\theta<\frac{\pi}{2}}{\frac{r}{2}<x<r}$. So it seems reasonable the average in "a" should be larger than the average in "b",

in part b area $\overset{\text{trapped}}{0<\theta<\frac{\pi}{4}}$ = area trapped $\frac{\pi}{4}<\theta<\frac{\pi}{2}$

Pg 384

S4.8

⑪ $[0, 2]$

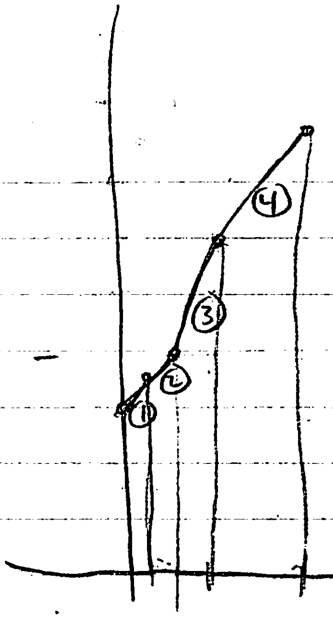
$f(0) = 3$

$f(\frac{1}{4}) = \frac{7}{2}$

$f(\frac{1}{2}) = 4$

$f(1) = 6$

$f(2) = 8$



assume each is a straight line.

$(0,3)$
$(\frac{1}{4}, \frac{7}{2})$ $\frac{3 - 3\frac{1}{2}}{0 - \frac{1}{4}} = \frac{-\frac{1}{2}}{-\frac{1}{4}} = 2 = m$

$y = 2x + 3$

$(\frac{1}{4}, \frac{7}{2})\ (\frac{1}{2}, 4)$ $y - 4 = \frac{1}{2}(x - \frac{1}{2})$

$\frac{4 - \frac{7}{2}}{\frac{1}{2} - \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$ $y = \frac{1}{2}x + \frac{15}{4}$

$\frac{1}{2} \cdot \frac{1}{2}(3 + 3\frac{1}{2}) +$

$(\frac{1}{2}, 4)$
$(1, 6)$ $\frac{6-4}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4 = m$

$y - 6 = 4(x - 1)$

$y = 4x + 2$

$(1, 6)$
$(2, 8)$ $\frac{8-6}{2-1} = 2 = m$

$y - 6 = 2(x - 1)$

$y = 2x + 4$

$$\int_0^{\frac{1}{4}} 2x+3\,dx + \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2}x + \frac{15}{4} + \int_{\frac{1}{2}}^{1} 4x+2 + \int_1^2 2x+4$$

$$x^2 + 3x\Big|_0^{\frac{1}{4}} + \frac{1}{4}x^2 + \frac{15}{4}x\Big|_{\frac{1}{4}}^{\frac{1}{2}} + 2x^2 + 2x\Big|_{\frac{1}{2}}^{1} + x^2 + 4x\Big|_1^2$$

4 80

$$\frac{\frac{1}{16} + \frac{3}{4}}{\frac{1}{4}} + \frac{\frac{1}{16} + \frac{15}{8} - \frac{1}{64} - \frac{15}{16}}{\frac{1}{4}} + \frac{2 + 2 - \frac{1}{2} - 1}{\frac{1}{2}} + \frac{\cancel{4} + 8 - 1 - \cancel{4}}{1}$$

8 − 3

$$\cancel{4}\,\frac{1+12}{\cancel{16}4} + 4\left(\frac{4 + 80 - 1 - 60}{64}\right) + 2\left(4 - \frac{3}{2}\right)$$

$$\frac{13}{4} + \frac{23}{16} + 5 + 6$$

$$\frac{42 + 23}{16} + \frac{176}{16} = \frac{241}{16} = 15.70625$$

Pg 384

Not finished

⑫ $at^2+bt+c=0$ has at most 1 root then $b^2-4ac \leq 0$.

a) if equation has 1 root $b^2-4AC=0$ if equation has no real roots $b^2-4ac<0$ ∴ if equation has at most 1 real root $b^2-4ac \leq 0$.

b)
$$h(t)=\int_A^B \left(t f(x)-g(x)\right)^2 dx$$

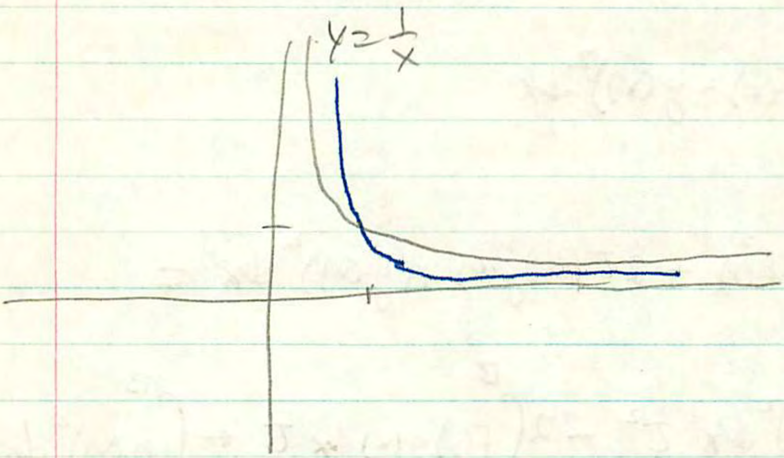
$$=\int_A^B t^2 f^2(x) - 2t f(x) g(x) + (g(x))^2 \, dx =$$

$$\underbrace{\int_A^B (f(x))^2 dx}_{a} \, t^2 - \underbrace{2\int_A^B f(x) g(x) dx}_{b} \, t + \underbrace{\int_A^B (g(x))^2 dx}_{c}$$

c) See b above

D)

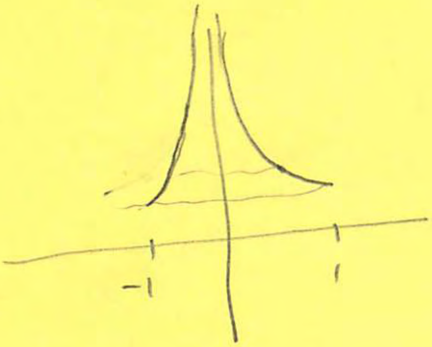
Absonote



Also consider

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -1-1=-2$$

Obviously area >0 so −2 is clearly wrong.



consider the
improper integrals;

$$\int_1^{\infty} \frac{1}{x}\,dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\,dx$$

Pg 285

we cannot integrate either of these as they stand. why not? FTC is defined over the closed interval [a, b] and ∞ is not an end point.

so we make a slight change:

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \qquad \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$\lim_{b\to\infty} \ln x \Big|_1^b \qquad \lim_{b\to 0} -\frac{1}{x}\Big|_1^b$$

$$\lim_{b\to\infty} \ln b \qquad \lim_{b\to\infty} -\frac{1}{b}+1$$

No Limit $\qquad$ 1

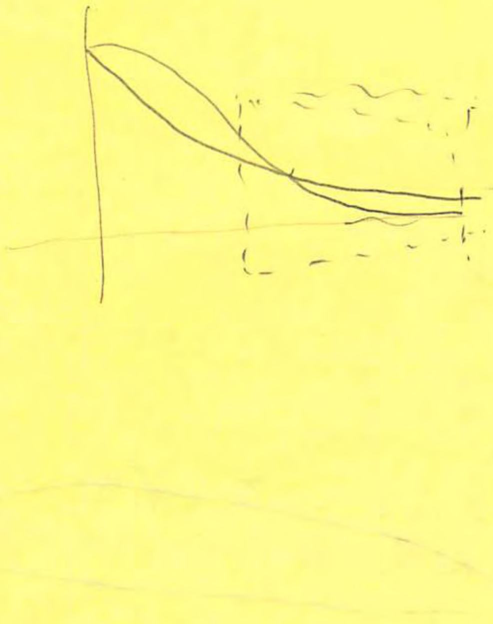
Consider $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$ like the previous expressions, this is also an improper integral because the expression following the integral sign becomes infinite at one or both of its end pts. or somewhere in between after converting to the limit form, if the limit exists then we say that the integral converges.

$Y_1 = e^{-x^2}$

$Y_2 = e^{-x}$

window x [0, 3] 1

y [-.5, 1] 1



Zoom memory STORE

Zoom Box

2 times

Zoom memory Recall

EXamples

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2}$$

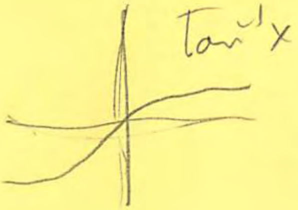
$$= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1$$

$$\int_{-\infty}^{0} \frac{dx}{\sqrt{1-x}} = \lim_{b\to-\infty}\int_b^0 \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{b\to-\infty} -2\sqrt{1-x}\Big|_b^0 = \lim_{b\to-\infty} 2\sqrt{1-b} - 2 \to +\infty$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} + \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}\Big|_0^b$$



$$\lim_{a\to\infty} -\tan^{-1}a + \lim_{b\to\infty} \tan^{-1}b$$

$$-\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$\pi$$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{\sqrt{x}}$$

$$= \lim_{b\to 0^+} 2\sqrt{x}\Big|_b^1 = \lim_{b\to 0^+} 2(1-b) = 2$$

$$\int_1^2 \frac{dx}{(x-2)^2} = \lim_{a\to 2^-} \int_1^a \frac{dx}{(x-2)^2}$$

$$= \lim_{a\to 2^-} \left(\frac{-1}{x-2}\right)\Big|_1^a = \lim_{a\to 2^-} \left(\frac{-1}{a-2} - 1\right) \to +\infty$$

on the other hand, if any part an integral or limit fails to exist, then the whole integral is said to diverge i.e. its value does not exist.

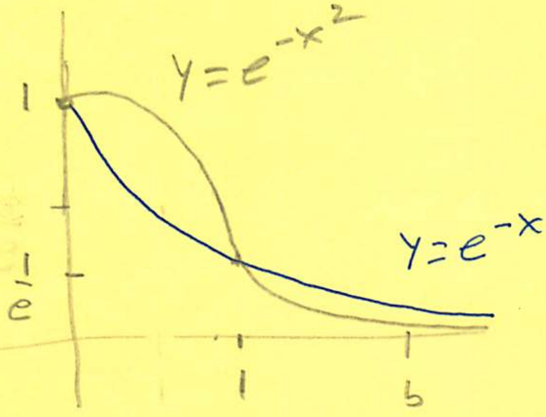
$$\int_0^\infty \cos x\,dx = \lim_{b\to\infty}\int_0^b \cos x\,dx$$

$$= \lim_{b\to\infty} \sin x\Big|_0^b$$

$$\lim_{b\to\infty} \sin b$$ No limit. DNE by oscillations. ∴ divergent.

another Technique to help with improper integrals is "TRAPPING" the integral below a known integral. Consider

$$\int_1^\infty e^{-x^2}dx \qquad \int_1^\infty e^{-x}dx$$



clearly $y=e^{-x}$ encloses more area than $y=e^{-x^2}$

$$\lim_{b\to\infty}\int_1^b e^{-x}dx \geq \int_0^\infty e^{-x^2}dx$$

$$\lim_{b\to\infty} -e^{-x}\Big|_1^b$$

$$\lim_{b\to\infty} -e^{-b}+e^{-1} = \frac{1}{e} \geq \int_0^\infty e^{-x^2}dx$$

thus $\int_1^\infty e^{-x^2}dx$ does converge to some value less than $\frac{1}{e}$.

~~HAND out GALAXIA~~

THE PROBLEM OF GALAXA : INFINITE AREA VERSUS FINITE VOLUME

By James McKim
University of Hartford, 1981

Galaxa was a nice Terran girl who at 2.587 years of age was looking forward to her first intelligence test. This would be her first chance to join the exalted ranks of the Perfs (those students who had obtained perfect scores on at least one examination). Most children take the test at age 2.589, but the Teachers had discovered that if a better-than-average student was given the test a few hours early, unusual and even ingenious solutions frequently resulted. Naturally, Galaxa had no idea what specific task she would be given, but she was reasonably sure she would need a good deal of mathematics and all of her skill with Enlargers.

The Enlarger was, of course, the most important and far-reaching invention of the twenty-second century (old style). Before Day 1 it was known that space was infinite but that matter was finite. What a dull universe it must have been! As any modern two-year-old knows, the Enlarger made infinite amounts of matter a reality. Of course, no Enlarger can take a finite chunk of matter and straightforwardly enlarge it to infinite size. What it can do is take any finite amount of matter, however small, and enlarge it to any *finite* size, however large. Further, while it can do nothing in no time, it can perform any simple task in as short a time as desired (much like modern Terran boys and girls). It required only a mildly clever computer to construct the universe's first infinitely long object–a wire–which was only one meter in length at the beginning of the experiment. The computer had the Enlarger double the first half-meter in half a second, then quadruple the next quarter meter in a quarter of a second, octuple the next eighth meter in an eighth of a second, and so on. Of course, by the end of one second, an infinitely long piece of wire was present in the universe, and the Enlarger, which had necessarily been attached to the growing part of the wire, was absent. (Modern Enlargers, and Terrans, are equipped with safety devices that allow them to do their work without leaving the universe.) Terran poets are fond of saying that the wire reached the "end of the universe" and that the fabled Enlarger is "out of this universe," and perhaps they are right.

The effects of the Enlarger and the advent of infinite matter were felt in all aspects of Terran life. Their present dating system, for example, is based on what appeared to be the spontaneous creation of matter. Then there was the unbalanced individual who, believing himself God, attempted to enlarge his brain infinitely in all directions at once. In the resulting explosion, parts of his head were exposed to his Enlarger longer than others. Travelers still return from that sector of space with tales of a huge planet which looks too much like an eye and another which looks too much like a nose. But we digress.

Galaxa faced the Teachers with a mixture of confidence, anticipation, and a dash of fear. She knew there was *no* real chance that she would fail the test; a century of genetic engineering had seen to that. But she could embarrass herself and her Teacher with a dismally low score. Further, she wouldn't have another chance to become a Perf until her next major exam almost 15.3 days away.

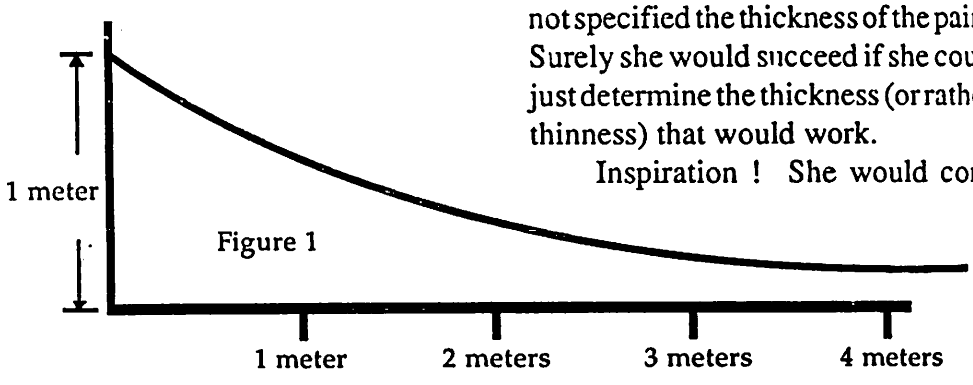


Figure 1

She examined the objects before her; π cubic meters of paint in a bucket, an Enlarger, a measuring stick, and a curiously shaped board of infinite length and no thickness, which looked something like figure 1. Galaxa knew that she might or might not need all these objects to accomplish her task.

A Teacher (*her* Teacher) spoke: "Your task is to paint the board . . ."

"Trivial," thought Galaxa, "I can make as much paint as I need with the Enlarger."

". . . without using the Enlarger on the paint," finished the Teacher. "Begin."

Galaxa spent a nanosecond in thought. If she couldn't use the Enlarger on the paint, then she didn't need any more paint. Hmm. Could it be that easy? She had heard that sometimes the Teachers' problems had obvious solutions that were overlooked by students seeking subtlety. She decided to try the obvious first. She told the paint to cover the first linear meter in half a second, the second meter in a quarter of a second, the third meter in an eight of a second, an so on.

She knew she had failed when the empty bucket returned *less* than a second later. The entire board could not have been covered in less than a second. It occurred to her that she had not specified the thickness of the paint. Surely she would succeed if she could just determine the thickness (or rather, thinness) that would work.

Inspiration ! She would compute the area of the board first. Her desired thickness would simply be π times the reciprocal of this area. (Remember the volume of the paint was π cubic meters.) Working furiously with the measuring stick she soon discovered that the curved portion of the board could be represented by :

$$y = \frac{1}{x}, \quad x \; \varepsilon \; [1, \infty)$$

And the area of the board was

$$A = \int_1^\infty \frac{1}{x}\, dx \; =$$

$$\lim_{k\to\infty} (\ln x)\Big|_1^k =$$

$$\lim_{k\to\infty} (\ln k) = \infty$$

Galaxa was close to despair! How could she hope to paint an infinite area with a finite amount of paint ? She knew that no matter how thin she made the paint, she could not cover the whole board. In fact, there would always be an infinite amount of area left! She wondered if the Teachers might have given her an impossible problem in order to see how long it would take her to discover its impossibility. Discouraged, she sat down on the bucket to think.

The bucket! One didn't need a bucket for modern paint. One simply told it to stay put. She then remembered some the special properties of the curve : $y = \frac{1}{x}$

Humming happily to herself, she began using the Enlarger on the bucket. When the Enlarger had finished, she went back over the infinitely long bucket shaping it with her strong Terran hands until it looked something like figure 2.

An ancient mathemagician might have gotten the sample shape by rotating: $y = \frac{1}{x}, \; x \; \varepsilon \; [1, \infty)$

about the x-axis. (This seems to be a skill lost to antiquity.) She knew that she had not changed the volume of her bucket, because...

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx =$$

$$V = \pi \lim_{k\to\infty} \left.\left(\frac{-1}{x}\right)\right|_1^k =$$

$$V = \pi \lim_{k\to\infty} \left(\frac{-1}{k} + 1\right) = \pi$$

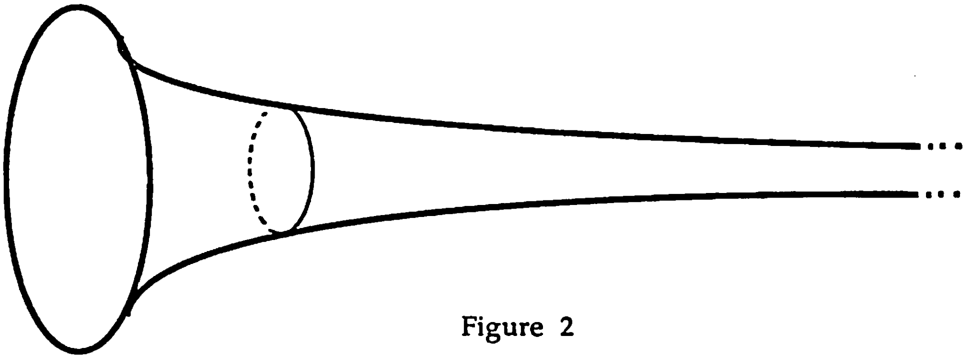


Figure 2

She ordered the paint into the bucket, and as expected, it fit exactly. She then carefully placed the board into the bucket, noticing that it would easily have fit into the top half. (Exactly how she accomplished this is left as an exercise for the alert reader.) No paint spilled out because the board had no volume (no thickness, remember?). Galaxa called the Teachers.

"You took a full millisecond longer than average," said the Teacher (*her* Teacher) sternly. Galaxa shivered. "But your technique was most unusual," the Teacher added, smiling. Galaxa smiled. "Most people apply the paint to the object, rather than the object to the paint," the Teacher continued. "No doubt you now see the easier way to accomplish the task. Neither the bucket nor the Enlarger are necessary."

Galaxa considered for a microsecond. "Of course! I should have made the paint thinner as I went along. For example, I could have used half the paint on the first linear meter, half of what was left on the next meter, half of *that* on the next meter and so on. The reason my method worked was that the board became very very quickly."

"Exactly!" said the oldest Teacher. (Galaxa estimated his age at 5.2. She thought he looked very distinguished with two of his baby teeth missing.) "Your score is relatively high, but finite."

Galaxa was pleased. The s of Aleph Null was awarded only for perfection. It had been out of her reach after her first abortive attempt to paint the board.

"Besides," she thought, wrinkling her nose, "the Perfs I know are just a bunch of tesseracts, anyway."

Then, too, there was always next time.

Improper integrals

10.6
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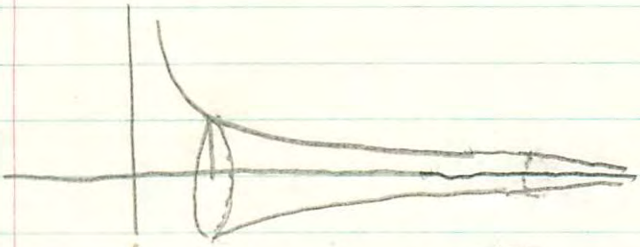
eg1

Graph $y = \frac{1}{x}$. Rotate about x axis and bounded on the left by $x = 1$. Determine the volume of this solid.



how about

$\pi\int\left(\frac{1}{x}\right)^2 dx$ as a start?

Since cross section is circle radius $\frac{1}{x}$.
what about bounds? from $x = 1$ to ......
$x = \infty$? $\pi\int_1^\infty \frac{dx}{x^2}$? this is an improper $\int$.

Remember FTC is defined over $[a, b]$!!

what about $\lim_{b\to\infty}\int_1^b \frac{dx}{x^2}$? much nicer

Since now we can use FTC.

$$\lim_{b\to\infty}\left(-\frac{\pi}{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{\pi}{b} + \pi\right) = \pi$$

eg2 Consider $\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx$ this is ALSO an improper integral. What do the these two integrals have in common? ......

⇒ the fctn which follows the integral sign becomes infinite at one or both of its limits of integration.

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eg 3 $\int_1^{\infty} \frac{dx}{x}$ and $\int_1^{\infty} \frac{dx}{x^2}$

$Y_1 = \frac{1}{x}$ $Y_2 = \frac{1}{x^2}$

Both approach x axis as $x \to \infty$

$\int_1^b \frac{1}{x} dx = \ln x \Big|_1^b = \ln b$ Now

$\lim_{b \to \infty} \ln b \Rightarrow$ increases indefinitely

we say $\lim_{b \to \infty} \int_1^b \frac{dx}{x} = \infty \therefore \int \frac{dx}{x} = \infty$ and this improper integral **diverges**.

however $\int_1^b \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^b = -\frac{1}{b} + 1$

$\lim_{b \to \infty} -\frac{1}{b} + 1 = 1$. Thus: This does have a definite limit, namely 1, as b increases indefinitely and we say that the integral from 1 to ∞ **converges** and that its value is 1.

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \to \infty}\left(1 - \frac{1}{b}\right) = 1.$$

Improper integrals such as $\int_{-\infty}^{b} f(x)\, dx$ are approached similarly.

Improper integrals such as $\int_{-\infty}^{\infty} f(x)\, dx$ are divided into $\int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx$ or some

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other convenient intervals, an improper integral such as $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent only if All of its parts are convergent.

eg 4 is $\int_0^{\infty} \cos x\,dx$ divergent or convergent?

$$\lim_{b\to\infty}\int_0^b \cos x\,dx = \lim_{b\to\infty}(\sin x)\Big|_0^b =$$

$\lim_{b\to\infty}(-\sin b)$. Since Sin b has no limit as $b\to\infty$ we say $\int_0^{\infty}\cos x\,dx$ is divergent.

we could say this limit does not exist by oscillations.

eg 5 even though we cannot find any simple expression for $\int_1^b e^{-x^2}dx$ (it is Not elementary) we can show that $\lim_{b\to\infty}\int_1^b e^{-x^2}dx$ exists and is finite. $\int^b e^{-x^2}dx$ represents the area between the x axis and the curve $y = e^{-x^2}$ between 1 and b. Clearly that area increases as x increases. There are two situations: either the integral becomes infinite or it becomes finite.

We shall show the 1st choice is impossible and conclude that the integral is finite ie it converges.

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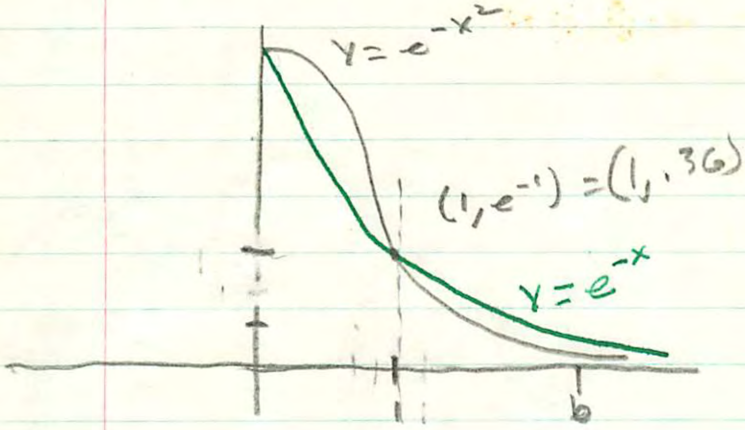
We do this by comparing the area under the curve $Y = e^{-x^2}$ with the area under the curve $Y = e^{-x}$



$$\int_1^b e^{-x}\,dx = -e^{-x}\Big|_1^b = e^{-1} - e^{-b}$$

$$\lim_{b \to \infty} e^{-1} - e^{-b} = \frac{1}{e}$$

Now Since e^{-x^2} is less than e^{-x} for $x > 1$

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = e^{-1}$$

so $\int_1^{\infty} e^{-x^2}\,dx$ converges to some value $\leq \frac{1}{e}$.

Now hand out "GALAXA" Ditto.

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① $$\int_1^{\infty} \frac{dx}{x^3} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to\infty}\left(-\frac{1}{2x^2}\Big|_1^b\right) = \lim_{b\to\infty} \frac{-1}{2b^2} + \frac{1}{2} = \frac{1}{2}$$

Converges to $\frac{1}{2}$

② $$\int_0^{\infty} \frac{dx}{x^3} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^3} + \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} =$$

$$\lim_{a\to 0^+} \frac{-1}{2x^2}\Big|_a^1 + \lim_{b\to\infty} \frac{-1}{2x^2}\Big|_1^b =$$

$$\lim_{a\to 0^+}\left(\frac{-1}{2a^2} + \frac{1}{2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2b^2} + \frac{1}{2}\right)$$

$$-\infty \qquad\qquad \frac{1}{2} \qquad \therefore \text{ diverges.}$$

③ $$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b =$$

$$\lim_{b\to\infty}\left(-e^{-b} + 1\right) = 1$$

④ $\int_0^{\infty} e^{-x}\sin 3x\,dx =$ 1st we'll set the integral then put in the limits etc.

$$\begin{array}{ccc} e^{-x} & + & \sin 3x \\ -e^{-x} & - & -\frac{1}{3}\cos 3x \\ e^{-x} & + & -\frac{1}{9}\sin 3x \end{array} \quad \int = -\frac{1}{3}e^{-x}\cos 3x - \frac{1}{9}e^{-x}\sin 3x - \frac{1}{9}\int e^{-x}\sin 3x$$

$$\frac{10}{9}\int = \text{etc}$$

$$\int e^{-x}\sin 3x\,dx = \frac{9}{10}\left[-\frac{1}{3}e^{-x}\cos 3x - \frac{1}{9}e^{-x}\sin 3x\right] =$$

$$\frac{-3}{10}e^{-x}\cos 3x - \frac{1}{10}e^{-x}\sin 3x\Big|_0^b$$

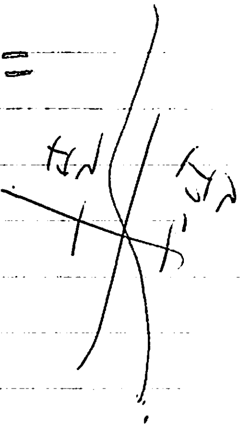
Now $$\lim_{b\to\infty}\left[\frac{-3\cos 3b}{10e^b} - \frac{\sin 3b}{10e^b}\right]\Big|_0^b = \frac{3}{10} + 0 = \frac{3}{10}$$

⑤ $\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^4}$ $\int \frac{x\,dx}{1+x^4}$ let $x^2 = u$
$2x\,dx = du$

$\frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u \Rightarrow \frac{1}{2}\tan^{-1}x^2$

$\lim_{a\to-\infty} \frac{1}{2}\tan^{-1}x^2\Big|_a^0 = \lim_{a\to\infty}\left(\frac{1}{2}\tan^{-1}0 - \frac{1}{2}\tan^{-1}a^2\right) =$
$0 - \frac{1}{2}\left(\frac{\pi}{2}\right) = -\frac{\pi}{4}$

$\lim_{b\to\infty} \frac{1}{2}\tan^{-1}x^2\Big|_0^b = \lim_{b\to\infty}\left(\frac{1}{2}\tan^{-1}b^2 - \frac{1}{2}\tan^{-1}0\right) =$
$\frac{1}{2}\frac{\pi}{2} - 0 = \frac{\pi}{4}$



$\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^4} = -\frac{\pi}{4} + \frac{\pi}{4} = 0$

⑥ $\int_1^{\infty} x^{-1.01}\,dx$ $\int x^{-1.01}\,dx = -\frac{1}{.01}x^{-.01}$

$\lim_{b\to\infty} -100x^{-.01}\Big|_1^b = \lim_{b\to\infty} -100\left(\frac{1}{b^{.01}} - 1\right) = -100(0-1) = 100$

$\lim_{b\to\infty} b^{.01-1} \to \infty$
$\therefore \lim_{b\to\infty}\frac{1}{b^{.01}} = 0$

⑦ $\int_0^1 x^{-1.01}\,dx$ $\int x^{-1.01}\,dx = -100\,x^{-.01}$

$\lim_{a\to0^+} -100x^{-.01}\Big|_a^1 = \lim_{a\to0^+}\left(-100 + 100a^{-.01}\right)$ diverges

$\lim_{a\to0} a^{.01} \Rightarrow 0$ because $a^{-.01} \to \infty$ as $a \to 0$
$\therefore \lim_{a\to0}\frac{1}{a^{.01}} \Rightarrow \infty$

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⑧ $\int_0^1 x^{-.99}dx$ $\qquad \int x^{-.99}dx = 100x^{.01}$

$\lim_{a\to 0^+} 100(x^{.01})\Big|_a^1 = \lim_{a\to 0^+} 100(1-a^{.01}) = 100$

⑨ $\int_0^1 x^{-1.01}$ $\qquad \int x^{-1.01}dx = -100\,x^{-.01}$

$\lim_{a\to 0^+} -100(x^{-.01})\Big|_a^1 = \lim_{a\to 0^+} -100(1-a^{-.01})$

Diverges since $\lim_{a\to 0^+} a^{-.01} \Rightarrow \infty$

⑩ $\int_0^e x\ln x\,dx$ $\qquad \begin{array}{ccc} \ln x & + & x \\ \frac{1}{x} & \leftarrow & \frac{1}{2}x^2 \end{array}$

we have improper $\int$ because $\ln x$ not defined for $x=0$.

$\frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$

$\lim_{a\to 0^+}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_a^e = \lim_{a\to 0^+}\left(\frac{1}{2}e^2\cdot 1 - \frac{1}{4}e^2 - \overset{\Downarrow}{\frac{1}{2}a^2\ln a} + \frac{1}{4}a^2\right)$

$\lim_{a\to 0^+} a^2\ln a = \lim_{a\to 0^+}\frac{\ln a}{\frac{1}{a^2}}$ $\quad \frac{-\infty}{\infty}$ use l'hapitals rule.

$\lim_{a\to 0^+}\frac{\frac{1}{a}}{\frac{-2}{a^3}} = \lim_{a\to 0^+} -\frac{1}{a}\,\frac{a^3}{2} = \lim_{a\to 0^+} -\frac{a^2}{2} = 0$ $\quad$ So

our integral has value $\frac{1}{2}e^2 - \frac{1}{4}e^2 = \frac{1}{4}e^2$

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⑪ $\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}$ $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$

$\lim_{a\to -1} \sin^{-1}x \Big|_a^0 + \lim_{b\to 1} \sin^{-1}x \Big|_0^b$

$\lim_{a\to -1}\left(\sin^{-1}0 - \sin^{-1}a\right) + \lim_{b\to 1}\left(\sin^{-1}b - \sin^{-1}0\right) =$

$-\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$

⑫ $\int_0^1 \frac{dx}{\sqrt{1-x}}$ let $u = 1-x$, $du = -dx$ $-\int \frac{du}{u^{\frac12}} = -2u^{1/2} =$

$\lim_{b\to 1} -2(1-x)^{\frac12}\Big|_0^b = \lim_{b\to 1} -2(1-b)^{1/2} + 2(1) = 2$

★ ⑬ $\int_0^4 \frac{dx}{\sqrt[3]{x-2}}$ let $u = x-2$, $du = dx$ $\int_{-2}^{2}\frac{du}{u^{1/3}} = \frac{3}{2}u^{2/3}\Big|_{-2}^{2} =$

$x=0$ $u=-2$
$x=4$ $u=2$

$\frac{3}{2}(2)^{2/3} - \frac{3}{2}(-2)^{2/3} = 3(2)^{1/3} - 3(2)^{1/3} = 0$

⑭ $\int_0^{\infty}\frac{dx}{\sqrt{x+1}} = \lim_{b\to\infty} 2\sqrt{x+1}\Big|_0^b = \lim_{b\to\infty}\left(2\sqrt{b+1} - 2\right)$ Diverges.

⑮ $\int_0^{\infty}(1-x)e^{-x}dx$ $u = 1-x$ $v = -e^{-x}$
$du = -dx$ $dv = e^{-x}$

$-(1-x)e^{-x} - \int e^{-x}dx = (x-1)e^{-x} + e^{-x} = xe^{-x}$

$\lim_{b\to\infty} xe^{-x}\Big|_0^b = \lim_{b\to\infty}\left(be^{-b} - 0\right) = \lim_{b\to\infty}\frac{b}{e^b} =$

$\frac{\infty}{\infty}$ l'hopital's Rule

$\lim_{b\to\infty}\frac{1}{e^b} = 0$

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16 $\int_0^{\infty} x^{-1/3}\,dx$ Neither is proper.

$$\int x^{-1/3}\,dx = \frac{3x^{2/3}}{2}$$

$$\lim_{\substack{a\to 0^+\\ b\to\infty}} \frac{3}{2}x^{2/3}\Big|_a^1\Big|_1^b = \lim_{\substack{a\to 0^+\\ \boxed{b\to\infty}}} \frac{3}{2}(\cancel{1}-a^{2/3}) + \frac{3}{2}(b^{2/3}\cancel{-1})$$

Diverges

⑰ $\int_0^1 \ln x\,dx$ $u=\ln x$ $\quad x$

$du=\frac{1}{x}dx \quad dv=dx$

$$x\ln x - \int dx = x\ln x - x$$

$$\lim_{a\to 0^+} x\ln x - x\Big|_a^1 = \lim_{a\to 0^+}(0-1-a\ln a+a) = -1$$

$\to 0$ via l'hospital's rule

$$\frac{\ln a}{\frac{1}{a}} \quad \frac{-\infty}{\infty}$$

$$\frac{\frac{1}{a}}{-\frac{1}{a^2}} = -a \to 0$$

⑱ $\int_1^{\infty} e^{-\sqrt[3]{x}}\,x^{-2/3}\,dx$

$$\lim_{a\to 0^+} a\ln a$$

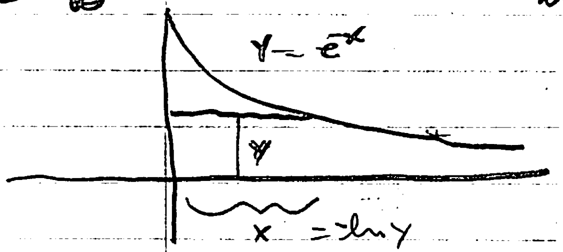
$$\lim_{a\to 0} \frac{\ln a}{\frac{1}{a}} \quad \frac{-\infty}{\infty}$$

$$\lim_{a\to 0} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$$

$$\lim_{a\to 0} -a = 0$$

using Shells

22b $\quad \ln y = -x$



$$-2\pi\int_0^1 y\ln y\,dy = \lim_{a\to 0^+} -2\pi\int_a^1 y\ln y\,dy$$

$u = \ln y \quad v = \frac{y^2}{2}$

$du = \frac{1}{y}dy \quad dv = y\,dy$

$$\lim_{a\to 0^+} -2\pi\left[\frac{1}{2}y^2\ln y\Big|_a^1 - \int_a^1 \frac{1}{2}y\,dy\right]$$

$$\lim_{a\to 0^+} -2\pi\left[\frac{1}{2}a^2\ln a - \frac{1}{4}y^2\Big|_a^1\right]$$

$$\lim_{a\to 0^+} -2\pi\left[\underbrace{\frac{1}{2}a^2\ln a}_{} - \frac{1}{4} + a\right] = \frac{\pi}{2}$$

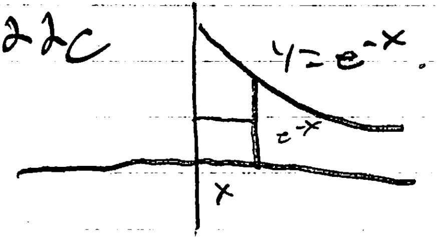
$$\lim_{a\to 0^+} a^2\ln a =$$

$$\lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a^2}} \quad \frac{-\infty}{\infty}$$

$$\lim_{a\to 0^+} \frac{\frac{1}{a}}{\frac{-2}{a^3}}$$

$$\lim_{a\to 0^+} \frac{a^2}{2} = 0.$$

22c



$$2\pi\int_0^\infty x e^{-x}\,dx = \lim_{b\to\infty} 2\pi\int_0^b x e^{-x}\,dx$$

$u = x \quad v = -e^{-x}$

$du = dx \quad dv = e^{-x}dx$

$$\lim_{b\to\infty} 2\pi\left[-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx\right]$$

$$\lim_{b\to\infty} 2\pi\left[-be^{-b} - e^{-x}\Big|_0^b\right]$$

$$\lim_{b\to\infty} 2\pi\left[-be^{-b} - e^{-b} + 1\right]$$

$$\lim_{b\to\infty} 2\pi\left[-e^{-b}(b+1) + 1\right]$$

$$\lim_{b\to\infty} \frac{b+1}{e^b} \quad \frac{\infty}{\infty} = \lim_{b\to\infty} \frac{1}{e^b} \Rightarrow 0$$

$$\lim_{b\to\infty} 2\pi\left[-e^{-b}(b+1) + 1\right] = 2\pi$$

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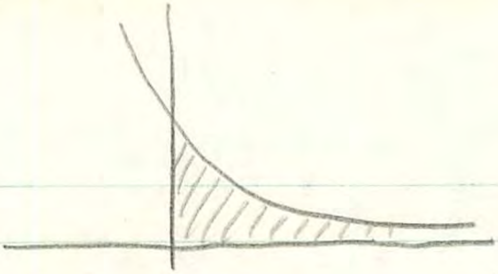
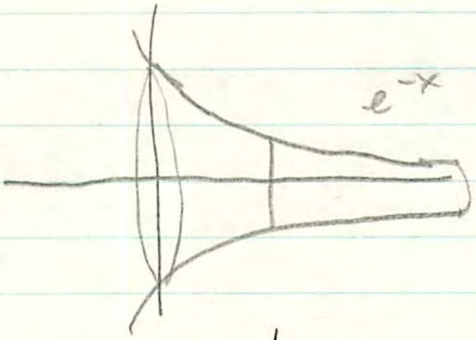


a) $\int_0^{\infty} e^{-x}\,dx \quad \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b =$

$\lim_{b\to\infty} \left(-e^{-b}+1\right) = 1$

b)



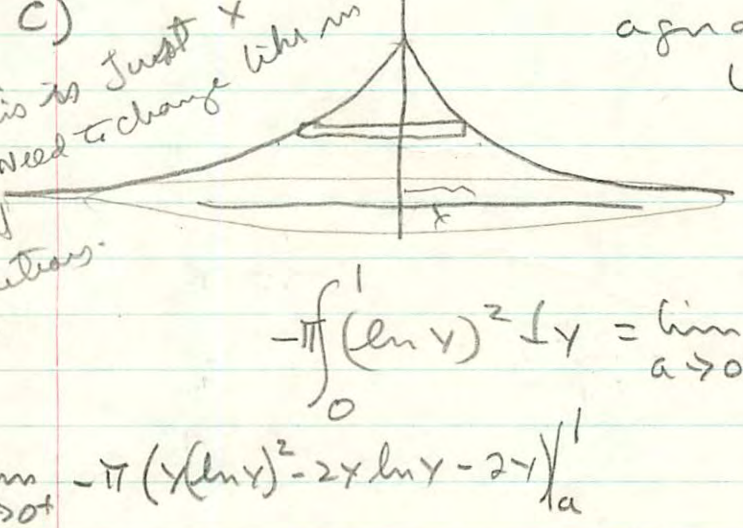
$\text{radius}^2 = e^{-2x}$

$\pi\int_0^{\infty} e^{-2x}\,dx$

$\lim_{b\to\infty} \pi\int_0^b e^{-2x}\,dx = \lim_{b\to\infty} -\frac{\pi}{2}e^{-2x}\Big|_0^b = \lim_{b\to\infty}\left(-\frac{\pi}{2}e^{-2b}+\frac{\pi}{2}\right) = \frac{\pi}{2}$

c)

radius is just x
Don't need to change like in horizontal cross sections.



agn a circle radius; $y = e^{-x}$,

$\ln y = -x \quad x = -\ln y$
↑
Radius.

$-\pi\int_0^1 (\ln y)^2\,dy = \lim_{a\to 0^+} -\pi\int_a^1 (\ln y)^2\,dy$

$u = (\ln y)^2 \quad dy$

$du = \frac{2\ln y}{y}\,dv = dx$

$y(\ln y)^2 - 2\int \ln y\,dy$

$y(\ln y)^2 - 2y\ln y - 2y$

$\lim_{a\to 0^+} -\pi\left(y(\ln y)^2 - 2y\ln y - 2y\right)\Big|_a^1$

$\lim_{a\to 0^+} +\pi\left(a(\ln a)^2 - 2a\ln a - 2a\right) - \pi(0-0-2) = \boxed{2\pi}$

agn l'hopital's Rule,

$\lim_{a\to 0^+} \underline{a(\ln a)^2} = \lim_{a\to 0^+} \frac{(\ln a)^2}{\frac{1}{a}} \Rightarrow \lim_{a\to 0^+} \frac{(2\ln a)\frac{1}{a}}{-\frac{1}{a^2}} \Rightarrow -2\lim_{a\to 0^+} a\ln a \Rightarrow$

$-2\lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} \Rightarrow -2\lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = -2\lim_{a\to 0^+} -a = 0$

Both Parts are included

Pg 391

㉓ $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ let $\alpha = \min\{a^2, b^2, c^2\}$

then : $(a^2+x)(b^2+x)(c^2+x) \geq (x+\alpha^2)^3$ so

$$0 \leq \frac{1}{(a^2+x)(b^2+x)(c^2+x)} \leq \frac{1}{(x+\alpha^2)^3} \quad \text{and}$$

$$0 \leq \frac{1}{\sqrt{\quad}} \leq \frac{1}{(x+\alpha^2)^{3/2}} .$$

$$\lim_{b\to\infty} \int_0^b \frac{dx}{(x+\alpha^2)^{3/2}} = \lim_{b\to\infty} \left(\frac{-2}{(x+\alpha)^{\frac{1}{2}}}\right)\Bigg|_0^b = \lim_{b\to\infty} \frac{-2}{(b+\alpha)^{\frac{1}{2}}} \Rightarrow 0$$

∴ original integral converges since this larger, similar integral converges

㉔ $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx$ $x \neq 0$ $f(0)=1$

Pg 391

㉗ $\int_0^{\infty} e^{-\tau}\tau^{-\frac{1}{2}}\,d\tau$

$$\lim_{a\to 0}\int_a^1 e^{-\tau}\tau^{-1/2}\,d\tau < \lim_{a\to 0}\int_a^1 \tau^{-1/2}\,d\tau$$

Since $e^{-\tau}$ in any Positive interval is less than 1,

$$\lim_{a\to 0}\int_a^1 \tau^{-1/2}\,d\tau = \lim_{a\to 0} 2\tau^{\frac{1}{2}}\Big|_a^1 = \lim_{a\to 0} 2-2a^{1/2} = 2$$

$$\lim_{b\to\infty}\int_1^b e^{-\tau}\tau^{-1/2}\,d\tau = \lim_{b\to\infty}\int_1^b \frac{d\tau}{e^{\tau}\tau^{1/2}} < \lim_{b\to\infty}\int_1^b \frac{d\tau}{e^{\tau}}$$

$$\lim_{b\to\infty}\int_1^b e^{-\tau}\,d\tau = \lim_{b\to\infty} -e^{-\tau}\Big|_1^b = \lim_{b\to\infty}\left(-e^{-b}-e^{-1}\right) = \frac{1}{e}$$

Since All parts converge to values greater that the given integral we can say that

$\int_0^{\infty} e^{-\tau}\tau^{-1/2}\,d\tau$ converges,

Note: integration of the original by parts yields a continually more difficult integral which does NOT "make a circle" so it cannot be simplified,

This accounts for the reason for the question — ie only does it converge rather than "evaluate it".

Polar Coordinates

were used in 1ST Semester

during XMAS Vacations.

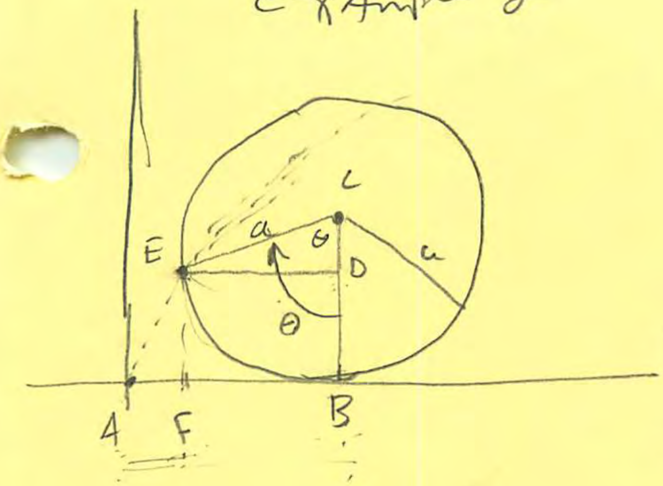
Lectures & exercises are

Supp ~~within~~ Ch APT 10 material.

are Now in beginning of

Chapt 7 & 8 Lecture Sections.

Example 2



$\overline{AB} = \overset{\frown}{EB}$

$EB = a\theta$ (radius)(angle)

$\therefore \overline{AB} = a\theta$

$\theta = 0 \quad x = 0 \quad y = 0$

already done

$AF = AB - \boxed{FB}$ same

$AF = a\theta - \boxed{ED}$

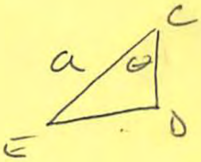
$a\sin\theta$

$AF = a\theta - a\sin\theta = x$ value.

$EF = CB - CD$

CB is radius a.

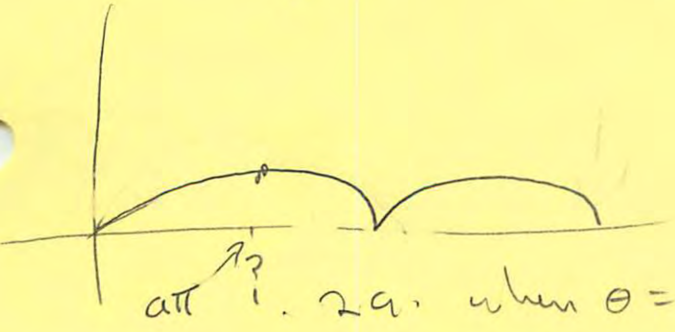
CD is $a\cos\theta$



$EF = a - a\cos\theta. = y$ value.

Thus

$$\begin{cases} x = a\theta - a\sin\theta \\ y = a - a\cos\theta \end{cases}$$



$a\pi$? . $2a$. when $\theta = 180° = \pi$

Show how Cycloid equations were developed. Point out the "convenience" of defining a set of parametric equations were one is $x = t$ as in eg following eg2 pg 398 Handout wanted DITTO

Pg 402

① $x = 2\tau + 1 \quad y = \tau - 1$

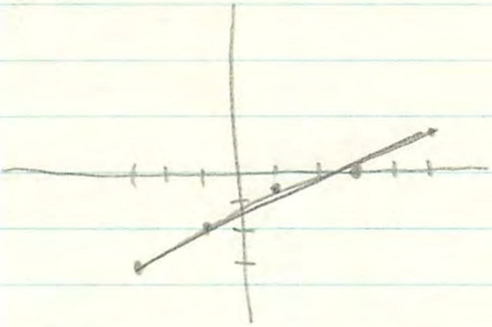
τ	-2	-1	0	1	2
x	-3	-1	1	3	5
y	-3	-2	-1	0	1

$x = 2(y+1) + 1$

$x = 2y + 2 + 1$

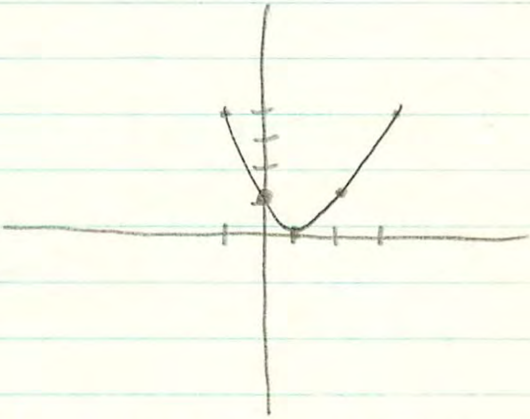
$2y = x - 3$

$y = \frac{1}{2}x - \frac{3}{2}$



② $x = \tau + 1 \qquad y = \tau^2$

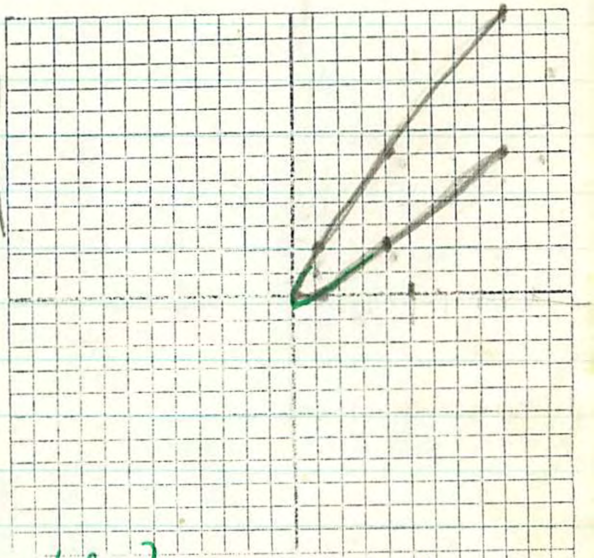
τ	-2	-1	0	1	2
x	-1	0	1	2	3
y	4	1	0	1	4



$y = (x-1)^2$

③ $x = \tau^2 \quad y = \tau^2 + \tau$

τ	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	6	2	0	0	2	6	12



$y = x + \sqrt{x}$

$y - x = \sqrt{x}$

$y^2 - 2xy + x^2 = x$

$y^2 + x^2 - 2xy - x = 0$

Is this possible?

Yes The curve is just Not a fctn.

$\tau = -\frac{1}{2} \quad x = \frac{1}{4}$

$y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

Pg 402

④

a) $\begin{cases} x = \tau^7 + \tau^2 + 1 \\ y = 2\tau^6 + 3\tau + 1 \end{cases}$

$\tau = -1$	$\tau = 0$	$\tau = 1$
$x = 1$	$x = 1$	$x = 3$
$y = 0$	$y = 1$	$y = 6$

b) I can'T eleminate τ

c) $\tau \to \infty$ $\quad \frac{y}{x}$ $\quad \lim_{\tau \to \infty} \frac{2\tau^6 + 3\tau + 1}{\tau^7 + \tau^2 + 1} = \lim_{\tau \to \infty} \frac{\frac{2}{\tau} + \frac{3}{\tau^6} + \frac{1}{\tau^7}}{1 + \frac{1}{\tau^5} + \frac{1}{\tau^7}} = 0$

D) $\frac{dx}{dT} = 7\tau^6 + 2\tau \Big|_{\tau=1} = 9$ $\qquad \frac{dy}{dx} = \frac{15}{9} = \frac{5}{3}$

$\frac{dy}{dT} = 12\tau^5 + 3 \Big|_{\tau=1} = 15$

⑤ $x = 2\cos\theta \qquad y = 3\sin\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{2\pi}{4}$	2π
x	2	1,414	0	-1,414	-2	-1,414	0	1,414	2
Y	0	2,121	3	+2,121	0	-2,121	-3	-2,121	0

$x^2 = 4\cos^2\theta \qquad y^2 = 9\sin^2\theta$

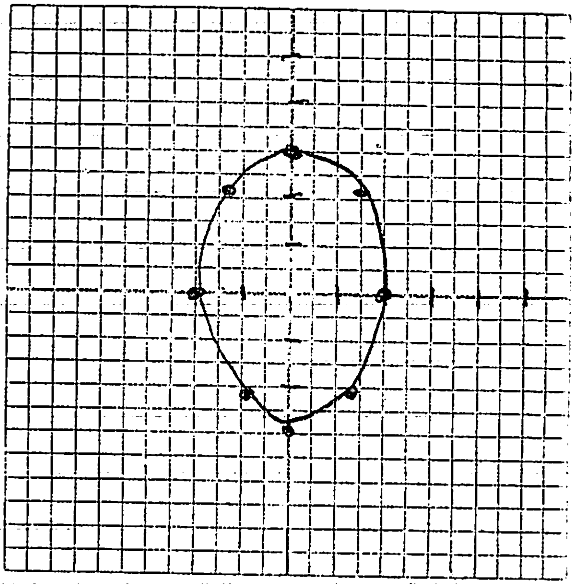
$9x^2 = 36\cos^2\theta \qquad 4y^2 = 36\sin^2\theta$

$\frac{9x^2}{36} = \cos^2\theta \qquad \frac{4y^2}{36} = \sin^2\theta$

$$\frac{9x^2}{36} + \frac{4y^2}{36} = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

ellipse



e) No it is Just a Parameter

Pg 402

⑥ a) $Y = x^3 - x^2$ b) $x = e^Y$

$$\begin{cases} x = \tau \\ Y = \tau^3 - \tau^2 \end{cases} \qquad \begin{cases} Y = \tau \\ x = e^\tau \end{cases}$$

c) $Y = x \sin x$ D) $x = \sin 3Y$

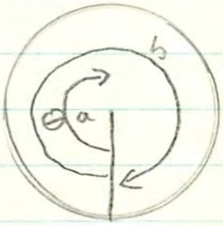
$$x = \tau \qquad Y = \tau \sin \tau \qquad \begin{cases} Y = \tau \\ x = \sin 2\tau \end{cases}$$

⑦ a $\theta = 180 = \pi$

b $\theta = 360 = 2\pi$

c $\theta = 360 + 180 = 540 = 3\pi$

D $\theta = 360 + 360 = 720 = 4\pi$

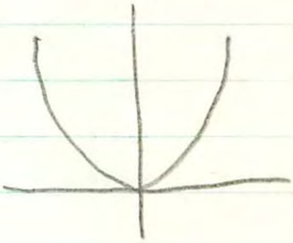


⑧ $$\begin{cases} x = (V_0 \cos \alpha)\tau \\ Y = (V_0 \sin \alpha)\tau - 16\tau^2 \end{cases}$$

$$\tau = \frac{x}{V_0 \cos\alpha} \qquad Y = V_0 \sin\alpha \cdot \frac{x}{V_0 \cos\alpha} - \frac{16\, x^2}{V_0 \cos\alpha}$$

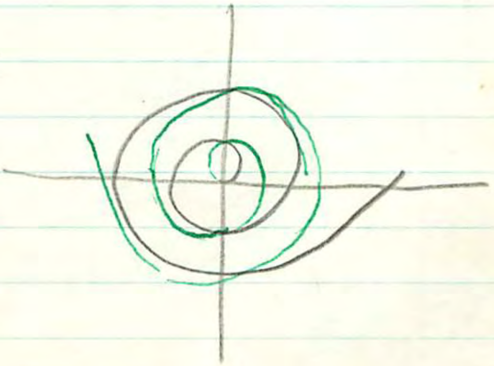
$$Y = x \tan\alpha - \frac{16x^2}{V_0^2 \cos^2\alpha}$$

⑨ $$\begin{cases} x = e^\tau \\ Y = e^{2\tau} \end{cases} \qquad x^2 = e^{2\tau} \quad \therefore \quad Y = x^2$$



⑩ $$\begin{cases} r = 3\tau \\ \theta = 6\tau \end{cases} \qquad \theta = 2r$$

as θ increases r increases
as θ decreases (neg) r still
increases (Absolute value)



/

⑪

$$x = \tau^5 + \sin 2\pi\tau$$
$$y = \tau + e^\tau$$

$$\frac{dx}{d\tau} = 5\tau^4 + 2\pi \cos 2\pi\tau \Big|_{\tau=1} = 5 + 2\pi$$

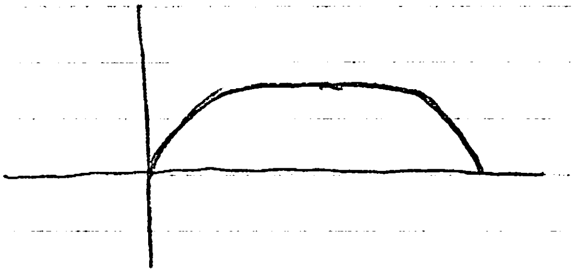
$$\frac{dy}{d\tau} = 1 + e^\tau \Big|_{\tau=1} = 1 + e$$

$$\frac{dy}{dx} = \frac{4 + e^\tau}{5 + 2\pi}$$

b) can'T be done.

? ⑫

$$\int_0^{2\pi a} y\,dx$$



$$x = a\theta - a\sin\theta$$
$$dx = a - a\cos\theta \; d\theta$$
$$y = a - a\cos\theta$$

$$y = 0 \Rightarrow \theta = 0$$
$$y = 2\pi a$$

$$2\pi = 1 - \cos\theta$$
$$-2\pi + 1 = \cos\theta$$
$$\theta = \cos^{-1}(1 - 2\pi)$$

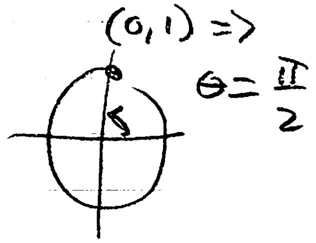
$$\int_0^{\cos^{-1}(1-2\pi)} (a - a\cos\theta)^2 \, d\theta \qquad \frac{1}{3}(a - a\cos\theta)^3 (a\sin\theta) \Big|_0^{\cos^{-1} 1 - 2\pi}$$

⑬ $r = 1 + \cos\theta$ $\quad x = r\cos\theta$ $\quad$ but $r = 1 + \cos\theta$ so
$\quad y = r\sin\theta$

a)
$$x = \overset{u}{(1+\cos\theta)}\overset{v}{\cos\theta}$$
$$y = (1+\underset{u}{\cos\theta})\underset{v}{\sin\theta}$$

$$\frac{dy}{d\theta} = -\sin^2\theta + \cos\theta + \cos^2\theta$$

$$\frac{dx}{d\theta} = -2\cos\theta\sin\theta - \sin\theta$$



b)
$$\frac{dy}{dx} = \left.\frac{\cos^2\theta + \cos\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta}\right|_{\substack{x=0\\ y=1\\ \theta=\frac{\pi}{2}}} = \frac{0+0-1}{-1-0} = 1$$

c) Slope of tangent near $(0,0) = (x,y)$ is when $\theta \to \pi^-$ (approaching from less than π keeps us above the x axis.

$$\lim_{\theta\to\pi^-} \frac{\cos^2\theta + \cos\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta} \quad \frac{1-1+0}{0-0} \quad \frac{0}{0}$$

apply l'hopital's rule

$$\lim_{\theta\to\pi^-} \frac{-2\cos\theta\sin\theta - \sin\theta - 2\sin\theta\cos\theta}{-\cos\theta + 2\sin^2\theta - 2\cos^2\theta} = \frac{0-0-0}{1+0-2} = 0$$

So slope approachs zero as cardioid nears the origin. or as $\theta \to \pi^-$

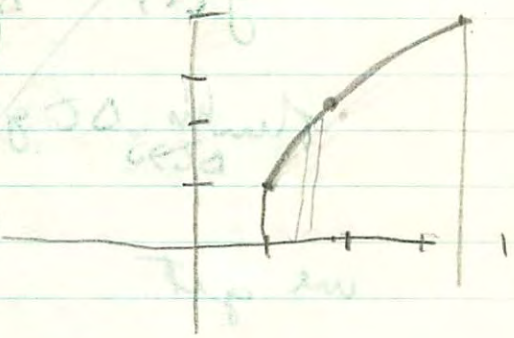
⑭ $x = \tau^2 + e^\tau \quad y = \tau + e^\tau \quad \tau \in [0, 1]$

$\tau = 0 \quad \tau = \frac{1}{2} \quad \tau = 1$

$x = 1 \quad x = \frac{1}{4} + e^{\frac{1}{2}} \quad x = 1 + e$

$y = 1 \quad y = \frac{1}{2} e^{\frac{1}{2}} \quad y = 1 + e$

$e^{\frac{1}{2}} = 1.65 \quad (1.9, 2.1) \quad (3.72, 3.72)$



$\frac{dy}{d\tau} = 1 + e^\tau$

$\frac{dx}{d\tau} = 2\tau + e^\tau$

$\frac{dy}{dx} = \frac{1 + e^\tau}{2\tau + e^\tau}\Big| = \frac{1+1}{0+1} = 2$

$\left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\} \tau = 0$

$\int_{x=a}^{x=b} y\,dx$ change everything to Parameter.

d) $\int_1^{e+1} \tau + e^\tau \, dx =$

$x = \tau^2 + e^\tau$

$dx = 2\tau + e^\tau \, d\tau$

$\int_0^1 (\tau + e^\tau)(2\tau + e^\tau)\, d\tau = \int_0^1 2\tau^2 + 3\tau e^\tau + e^{2\tau} \, d\tau =$

$\frac{2}{3}\tau^3 + \frac{1}{2} e^{2\tau} + 3\tau e^\tau - 3\int_0^1 e^\tau \, d\tau \qquad \begin{matrix} 3\tau & e^\tau \\ 3 & e^\tau \end{matrix}$

$\frac{2}{3}\tau^3 + \frac{1}{2} e^{2\tau} + 3\tau e^\tau - 3e^\tau \Big|_0^1 =$

$\frac{2}{3} + \frac{1}{2} e^2 + \cancel{3e} - \cancel{3e} - 0 - \frac{1}{2} + 3 \qquad \frac{7}{2}$

$\frac{4}{6} - \frac{21}{6} = \frac{17}{6} \qquad \boxed{\frac{1}{2} e^2 + \frac{19}{6}}$

Since we know (for example)

$$g'(\tau) = \lim_{\Delta\tau\to 0} \frac{g(\tau+\Delta\tau)-g(\tau)}{\Delta\tau}$$

Then $\lim_{\Delta\tau\to 0} \Delta\tau\, g'(\tau) = \lim_{\Delta t\to 0} g(\tau+\Delta\tau)-g(\tau)$

we get

$$\sqrt{(\Delta\tau)^2(g'(\tau))^2}$$

---

consider the interval $[\tau_i \quad \tau_i+\Delta\tau]$

$\exists\ T_i \in$ interval $\ni$

$$g(\tau_i+\Delta\tau) - g(\tau_i) = g(T_i)(\tau_i+\Delta\tau-\tau_i)$$

$$= g(T_i)\,\Delta\tau$$

$$\therefore (\qquad)^2 = (g(T_i))^2\,\Delta\tau^2$$

Law of the mean says: $f'(\overline{X}) =$ ~~$\frac{f(x)-f(x+\Delta x)}{x - (x+\Delta x)}$~~

ie $\exists\ \overline{X} \in (x, x+\Delta x) \ni$

~~Thus $\Delta x\, f'(\overline{X}) = f(x) - f(x+\Delta x)$~~

$$f'(\overline{X}) = \frac{f(x+\Delta x)-f(x)}{x+\Delta x-x}$$

Thus

$$\Delta x\, f'(\overline{X}) = f(x+\Delta x)-f(x)$$

Law of the Mean (pg 172)

AKA Mean Value Thm.

AKA Mean Value Thm for Derivatives.

There is a $\bar{X} \in (a,b)$ such that the slope of the tangent at $(\bar{X}, f\bar{X})$ is parallel to the chord from $(a, f(a))$ to $(b, f(b))$ and is written

$$f'(\bar{X}) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Notice $\bar{X}$ Also notice no limit on right side of =.

~~Finally Did you notice $\bar{X} \in (a,b)$ is open interval~~?

Thus $\Delta x \, f'(\bar{X}) = f(x+\Delta x) - f(x)$

Now ....

Given the movement of a point in terms of τ as

$$\begin{cases} x = g(\tau) \\ y = h(\tau) \end{cases}$$

Let Point P be time a be τ_i and another point Q, at a later time be $\tau_i + \Delta\tau$

The chord from P to Q is ☆

①

Given: The movement of a point in terms of τ.

$$\begin{cases} x = g(\tau) \\ y = h(\tau) \end{cases}$$

Length of curve

we are at point P at time τ_i so we have coordinates $x = g(\tau_i)$ and $y = h(\tau_i)$ ie $(g(\tau_i), h(\tau_i)$

Then at $\Delta\tau$ seconds later we are at point Q

so $x = g(\tau_i + \Delta\tau)$ and $y = h(\tau_i + \Delta\tau)$

The straight line distance from P to Q is

★ $$\overline{PQ} = d = \sqrt{(g(\tau_i + \Delta\tau) - g(\tau_i))^2 + (h(\tau_i + \Delta\tau) - h(\tau_i))^2}$$

we digress a moment ....

The Law of the Mean (pg 172)

"There is a $\underline{\overline{X}} \in (a,b)$ such that the slope of the tangent at $(\underline{\overline{X}}, f(\underline{\overline{X}}))$ is parallel to the chord from $(a, f(a))$ to $(b, f(b))$ and is written:

$$f'(\underline{\overline{X}}) = \frac{f(x+\Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

① Notice $\underline{\overline{X}}$ ② Notice No Lim so $\Delta x \, f'(X) = f(x+\Delta x) - f(x)$.

Thus in ✱ above NoTice The 'T'

$$\overline{PQ} = d = \sqrt{\left(\Delta\tau \, g'('T')\right)^2 + \left(\Delta\tau \, h'('T')\right)^2}$$

$$= \Delta\tau \sqrt{\left(g'('T')\right)^2 + \left(h'('T')\right)^2}$$

Remember $\tau \le 'T' \le \tau + \Delta\tau$ (because Law of The mean here $X \in (a,b)$)

When we take the limit as $\Delta\tau \to 0$ 'T' becomes τ. and what does the "length of the chord" become? iT becomes just a point.

If we add up all these points ..... The $\Delta\tau$ becomes $d\tau$

$$\int_a^b \sqrt{\left(g'(\tau)\right)^2 + \left(h'(\tau)\right)^2} \, d\tau$$

we get the length of the curve

defined by $\begin{cases} x = g(\tau) \\ y = h(\tau) \end{cases}$!

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\left(\frac{dx}{dt}\right)^2\left(1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2\right)$$

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)\left(\frac{dx}{dt}\right)^2$$

what if we substitute $\frac{dy}{dt}$ for $h'(t)$ and $\frac{dx}{dt}$ for $g'(t)$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Leave Room for $\sqrt{\dot{x}^2+\dot{y}^2}$ →

Then what if we factor out $\left(\frac{dx}{dt}\right)^2$ . . .

$$\int_a^b \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,dt\cdot\frac{dx}{dt} =$$

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$ or

$$\int_a^b \sqrt{1+(y')^2}\,dx$$ or

$$\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$

we see that $\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ = length of the line.

Lets consider polar graphing (r, θ) *

$$x = \overset{u}{r}\,\overset{v}{\cos\theta} \qquad y = \overset{u}{r}\,\overset{v}{\sin\theta}$$

$$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta \qquad \frac{dy}{d\theta} = \sin\theta\frac{dr}{dt} + r\cos\theta$$

$$\left(\frac{dx}{dt}\right)^2 = \cos^2\theta\left(\frac{dr}{dt}\right)^2 - 2r\sin\theta\cos\theta\,d\theta + r^2\sin^2\theta$$

$$\left(\frac{dy}{d\theta}\right)^2 = \sin^2\theta\left(\frac{dr}{dt}\right)^2 + 2r\sin\theta\cos\theta + r^2\cos^2\theta$$

Here, θ is the parameter rather than t.

$$\int\sqrt{\begin{matrix}\cos^2\theta\left(\frac{dr}{d\theta}\right)^2 - 2r\sin\theta\cos\theta\,d\theta + r^2\sin^2\theta \\ \sin^2\theta\left(\frac{dr}{d\theta}\right)^2 + 2r\sin\theta\cos\theta\,d\theta + r^2\cos^2\theta\end{matrix}}\;d\theta$$

$$\int\sqrt{\frac{dr}{d\theta}^2 + r^2}\;d\theta.$$

$$\int\sqrt{\dot r^2 + r^2}\;d\theta$$

* r is a fctn of θ ie r is NOT a constant ∴ we must use product rule to get $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$.

Given $\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \end{cases}$

Find the length of the curve from $\theta = 0$ to $\theta = 2\pi$

$$\begin{cases} \dot{x} = -3\sin\theta \\ \dot{y} = 3\cos\theta \end{cases} \qquad \begin{matrix} (\dot{x})^2 = 9\sin^2\theta \\ (\dot{y})^2 = 9\cos^2\theta \end{matrix}$$

$$\int_0^{2\pi} \sqrt{9\sin^2\theta + 9\cos^2\theta}\; d\theta$$

$$\int_0^{2\pi} 3\, d\theta = 3\theta\Big|_0^{2\pi} = 6\pi$$

Is This reasonable? what was the original curve? a circle of radius 3. Circumference is $2\pi r = 6\pi$.

many times these "lengths of the line" become very complicated integrals. The good news is that many times we want the speed rather than the distance. Why is this so good? Speed is the derivative of distance and using the 2nd FTC

$$G(x) = \int_a^x f(t)\,dt$$

$$G'(x) = f(x)$$

If $S(x) = \int_a^x \sqrt{g'(t) + h'(t)}\,dt$

$$S'(x) = \sqrt{(g'(x))^2 + h'(x)^2}$$

P 6

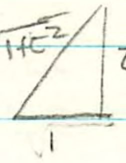
Example 1 Given path of Ball $\begin{cases} x = 32t \\ y = -16t^2 \end{cases}$ $\dot{x} = 32$

$\dot{y} = -32t$

To find distance traveled in 6 seconds.

$$\int_0^6 \sqrt{(32)^2 + 32t^2}\, dt$$

$$32\int_0^6 \sqrt{1+t^2}\, dt$$



$t = \tan\theta$

$dt = \sec^2\theta\, d\theta$

$$32\int_0^{\tan^{-1}6} \sec^3\theta\, d\theta.$$ etc.

eg #2 cycloid

find length of 1 arch

$[0, 2\pi]$

$\begin{cases} x = a\theta - a\sin\theta \\ y = a - a\cos\theta \end{cases}$

$\dot{x} = a - a\cos\theta$

$(\dot{x})^2 = a^2 - 2a^2\cos\theta + a^2\cos^2\theta$

$\dot{y} = a\sin\theta$

$(\dot{y})^2 = a^2\sin^2\theta$

$$\int_0^{2\pi} \sqrt{a^2 - 2a^2\cos\theta + \underbrace{a^2\cos^2\theta + a^2\sin^2\theta}_{a^2}}\, d\theta$$

$$\int_0^{2\pi} \sqrt{2a^2 - 2a^2\cos\theta}\, d\theta$$

$$a\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\, d\theta.$$

Note $\sin x = \sqrt{\frac{1}{2}(1-\cos 2x)}$

so $\sin\frac{A}{2} = \sqrt{\frac{1}{2}(1-\cos A)}$

$\sqrt{2}\sin\frac{A}{2} = \sqrt{1-\cos A}$

$$a\sqrt{2}\int_0^{2\pi} \sqrt{2}\sin\frac{\theta}{2}\, d\theta$$

$$2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

$$2a(-2)\cos\frac{\theta}{2}\Big|_0^{2\pi} = -4a(-1) + 4a = 8a.$$

There is a silver lining however. Generally we are not asked to find the distance along a curve. we are more concerned with the Speed along this path rather than the path itself!

Speed is the 1st derivative of distance so the FTC will be of use to us,

$$G(x) = \int_a^x f(t)\,dt$$

$$G'(x) = \frac{d\left(\int_a^x f(t)\,dt\right)}{dx} = f(x), \quad \text{so,}$$

if $S(x) = \int_a^{\tau} \sqrt{(g'(x))^2 + (h'(x))^2}\,dx$ then

Speed $= S'(\tau) = \frac{ds}{dt} = \sqrt{(g'(\tau))^2 + (h'(\tau))^2}$

~~which leads to~~

eg. again go back to the Ball thrown

$x = 32\tau$ $\quad \frac{dx}{dt} = 32$

$y = -16\tau^2$ $\quad \frac{dy}{dt} = -32\tau$

the speed of the ball is $\sqrt{32^2 + (32\tau)^2} = 32\sqrt{1+\tau^2}$. feet/sec.

~~we should expect to see some reference made to curves stated in polar coordinate.~~

↓

End of lecture

$\int_a^b \sqrt{1+(f'(x))^2}\,dx$ Length of the line.

but $f'(x) = \frac{dy}{dx}$

$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ normally written $\int_a^b ds$ because

Notice

$ds = \sqrt{(dx)^2+(dy)^2}$

other forms $\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ $\int_a^b \sqrt{\dot{x}^2+\dot{y}^2}\,d\tau$

$\int_a^b \sqrt{r^2+(r')^2}\,d\theta$

~~$y = \sin\theta$~~

~~$x = 1+\cos\theta$~~

~~$dx = -\sin\theta\,d\theta$~~

$f(\theta) = \cos\theta$

~~$f'(\theta) = -\sin\theta$~~

~~$\sqrt{1+\sin^2\theta}\;d\theta$~~

~~$\sqrt{\sin^2\theta + \cos^2\theta}\;d\theta$~~

Pg 403-411
10.9

The length of the curve $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ is

$$\int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta =$$

$$\int_\alpha^\beta \sqrt{r^2 + (r'(\theta))^2}\, d\theta.$$ if f' is assumed to be continuous.

In order to convince us of the validity of these formulas:

$(r, \theta) = (f(\theta), \theta)$ change to rectangular coordinates:

Prime (ie Y') means $\frac{dY}{dx}$. The dot (ie $\dot{Y}$) means WRT something other than x, usually time.

$$\begin{cases} x = \overset{r\ \cos\theta}{f(\theta)\cos\theta} \\ Y = \underset{r\ \sin\theta}{f(\theta)\sin\theta} \end{cases} \qquad \begin{aligned} \dot{x} &= -f(\theta)\sin\theta + f'(\theta)\cos\theta \\ \dot{Y} &= f(\theta)\cos\theta + f'(\theta)\sin\theta \end{aligned}$$

Now

$$\int_\alpha^\beta \sqrt{\dot{x}^2 + \dot{Y}^2}\, d\theta = \int_\alpha^\beta \sqrt{(-f(\theta)\overset{\sin\theta}{\,} + f'(\theta)\cos\theta)^2 + (f(\theta)\cos\theta + f'(\theta)\sin\theta)^2}\, d\theta$$

$$= \sin^2\theta f^2(\theta) \underline{- 2f'(\theta)f(\theta)\sin\theta\cos\theta} + (f'(\theta))^2\cos^2\theta + (f(\theta))^2\cos^2\theta + \underline{2f'(\theta)f(\theta)\cos\theta\sin\theta} + (f'(\theta))^2\sin^2\theta$$

$$= \sqrt{(f(\theta))^2\sin^2\theta + f(\theta)^2\cos^2\theta + (f'(\theta))^2\cos^2\theta + (f'(\theta))^2\sin^2\theta}$$

$$= \sqrt{f(\theta)^2(\sin^2\theta + \cos^2\theta) + f'(\theta)^2(\sin^2\theta + \cos^2\theta)} =$$

$$= \sqrt{f^2(\theta) + (f'(\theta))^2}$$

Pg 403-411

Now $\sin 2x = \sqrt{\frac{1}{2}(1-\cos x)}$

(we used $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ earlier) so

$\sqrt{2}\sin\frac{\theta}{2} = \sqrt{1-\cos\theta}$

$$a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$$

$$a\sqrt{2}\int_0^{2\pi}\sqrt{2}\sin\frac{\theta}{2}\,d\theta$$

$$2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta = 2a(-2)\cos\frac{\theta}{2}\Big|_0^{2\pi} =$$

$$-4a(-1)+4a(1) = 8a.$$

☆ if the fctn is not given parametrically we can always rewrite it so it is parametric ie $y=f(x) = \begin{cases} x=t \\ y=f(t) \end{cases}$

$\int_a^b \sqrt{1+(f'(t))^2}\,dt$ but t is just x so $dt = dx$

so $\int_a^b \sqrt{1+f'(x)^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ so

actually we do not need to change to a parameter.

eg 3 find the length of $y=x^3$ for $x \in [1,2]$.

$y' = 3x^2$

$$\int_1^2 \sqrt{1+9x^4}\,dx$$

unfortunately this is not elementary so we cannot evaluate this using FTC.

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

but $x_i - x_{i-1} = \Delta x$

so $$\frac{f(x_i) - f(x_{i-1})}{\Delta x} = f'(x)$$ by law of the mean

slope of chord [illegible]

= slope of ~~same~~ tangent line at $x, f(x)$

$.000000006512 \times 10^{-9}$ Pg 403-411
10.9

eg given $r = e^{-3\theta}$ $\theta \in [0, 2\pi]$ a spiral

the length can be found:

$$r' = \frac{dr}{d\theta} = -3e^{-3\theta} \quad \text{then}$$

$$\text{Arc length} = \int_{\alpha}^{\beta} \sqrt{r^2 + (r'\theta)^2}\, d\theta \Rightarrow$$

$$\int_0^{2\pi} \sqrt{e^{-6\theta} + 9e^{-6\theta}}\, d\theta = \int_0^{2\pi} \sqrt{10e^{-6\theta}}\, d\theta$$

$$\sqrt{10}\int_0^{2\pi} e^{-3\theta}\, d\theta = \frac{\sqrt{10}}{-3} e^{-3\theta}\Big|_0^{2\pi} =$$

$$\frac{\sqrt{10}}{-3}\left[e^{-6\pi} - 1\right] \doteq 1.054$$

Don't discuss memory aids. let those who read the sections get it.

Pg 410

① $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1} \quad x \in [1,2]$

$$S = \int_a^b \sqrt{1 + y'^2}\,dx \qquad y' = x^2 - \tfrac{1}{4}x^{-2}$$

$$x^2 - \frac{1}{4x^2}$$

$$(y')^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}$$

$$S = \int_1^2 \sqrt{1 + (x^2 - \tfrac{1}{4}x^{-2})^2}\,dx =$$

$$= \int_1^2 \sqrt{1 + x^4 - \tfrac{1}{2} + \tfrac{1}{16x^4}}\,dx = \int_1^2 \sqrt{x^4 + \tfrac{1}{2} + \tfrac{1}{16x^4}}\,dx$$

$$= \int_1^2 \sqrt{(x^2 + \tfrac{1}{4x^2})^2}\,dx = \int_1^2 x^2 + \frac{1}{4x^2}\,dx =$$

$$\frac{1}{3}x^3 - \frac{1}{4}\cdot\frac{1}{x}\Big|_1^2 = \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4} =$$

$$\frac{64 - 3 - 8 + 6}{24} = \frac{59}{24}$$

② $y = \frac{1}{2x^2} + \frac{x^4}{16} \quad x \in [2,3] \qquad y = \frac{1}{2}x^{-2} + \frac{x^4}{16}$

$$y' = -x^{-3} + \frac{1}{4}x^3$$

$$(y')^2 = x^{-6} - \frac{1}{2} + \frac{1}{16}x^6 \qquad \int_2^3 \sqrt{1 + (y')^2} =$$

$$1 + (y')^2 = x^{-6} + \frac{1}{2} + \frac{1}{16}x^6 = (x^{-3} + \tfrac{1}{4}x^3)^2 \Rightarrow$$

$$\int_2^3 x^{-3} + \frac{1}{4}x^3\,dx = -\frac{1}{2}x^{-2} + \frac{1}{16}x^4\Big|_2^3 =$$

$$-\frac{1}{18} + \frac{81}{16} + \frac{1}{8} - 1 = \frac{-8 + 729 + 18 - 144}{144} = \frac{595}{144}$$

$$= \int_{T=a}^{T=b} \sqrt{x'^2 + y'^2}\, dT$$

pg 410

③ find $\frac{ds}{dt}$ at $t = 0, 1, 2$.

pg 408 $\frac{ds}{dt} = 32\sqrt{1+t^2}$ feet/sec.

$\left.\frac{ds}{dt}\right|_{t=0} = 32$ $\left.\frac{ds}{dt}\right|_{t=1} = 32\sqrt{2}$ $\left.\frac{ds}{dt}\right|_{t=2}$ $32\sqrt{5}$.

initial velocity

④

$P = (\cos t + t\sin t, \sin t - t\cos t)$

find dist from $t=1$ to $t=\pi$

$x = \cos t + t\sin t$

$\frac{dx}{dt} = -\sin t + \sin t + t\cos t = \underline{\underline{t\cos t}}$

$y = \sin t - t\cos t$

$\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$.

$$\int_1^{\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = \int_1^{\pi} t\,dt =$$

$$\left.\frac{1}{2}t^2\right|_1^{\pi} = \frac{\pi^2}{2} - \frac{1}{2}.$$

Pg 410

⑤ (t^2, t^3) $\qquad x = t^2 \qquad y = t^3$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2$

$t=1 \to t=2$

$$s = \int_1^2 \sqrt{4t^2+9t^4}\,dt = \int_1^2 t\sqrt{4+9t^2}\,dt$$

let $9t^2+4 = u$

$18t\,dt = du$

$t=1 \quad u=13$

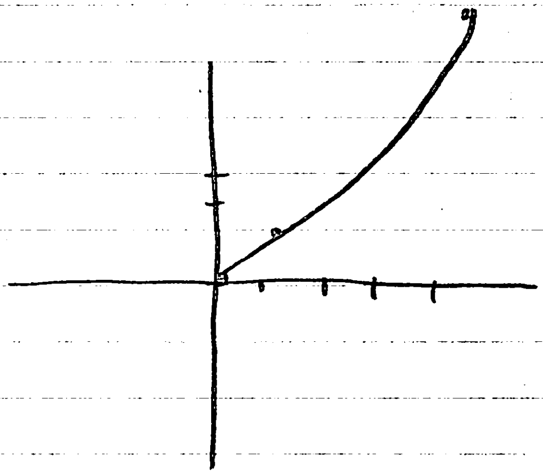
$t=2 \quad u=40$

$$\frac{1}{18}\int u^{1/2}\,du =$$

$$\frac{1}{\cancel{18}_{9}} \cdot \frac{\cancel{2}}{3} u^{3/2}\Big|_{13}^{40} =$$

a) $\frac{1}{27}\left((40)^{3/2} - 13^{3/2}\right) = \frac{1}{27}(252.98 - 46.872)$

$\frac{1}{27}(206.11) = 7.6337$



b) $\frac{ds}{dt} = t\sqrt{4+9t^2}$

c) $(1,1) \quad (4,8)$

d) $y = x\sqrt{x} = x^{3/2}$

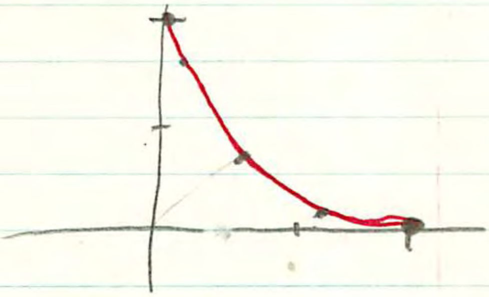
Pg 410

⑥

$(\cos^3\tau, \sin^3\tau)$ $\tau = 0 \Rightarrow \tau = \frac{\pi}{2}$

a)

τ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x	1	.65	.35	.125	0
y	0	.125	.35	.65	1



$x = \cos^3\tau$

$\frac{dx}{d\tau} = -3\cos^2\tau\sin\tau$

$\frac{dy}{d\tau} = 3\sin^2\tau\cos\tau$

$$\int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2\tau\sin\tau)^2 + (3\sin^2\tau\cos\tau)}\,dt$$

$$\sqrt{9\cos^4\tau\sin^4\tau + 9\sin^4\tau\cos^4\tau}$$

$$\sqrt{9\cos^2\tau\sin^2\tau\,(\cos^2\tau + \sin^2 t)} = 3\cos\tau\sin\tau$$

$$3\int_0^{\frac{\pi}{2}} \cos t\sin\tau\,dt$$

let $u = \sin\tau$
$du = \cos t\,dt$
$\tau = 0 \Rightarrow u = 0$
$\tau = \frac{\pi}{2} \Rightarrow u = 1$

b) $3\int_0^1 u\,du = \frac{3}{2}u^2\Big|_0^1 = \frac{3}{2}.$

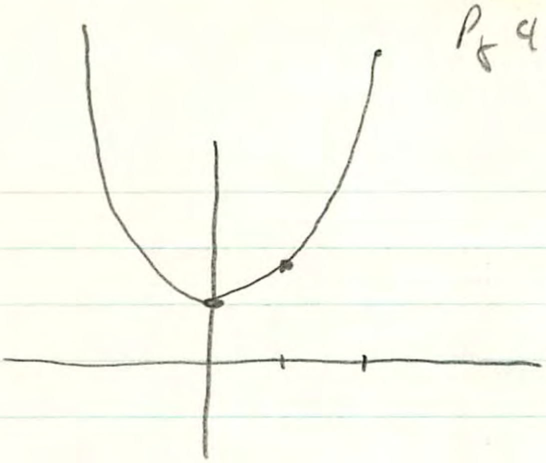
c) $x = \cos^3\tau$ $y = \sin^3\tau$

$x^{\frac{1}{3}} = \cos\tau$ $y^{\frac{1}{3}} = \sin\tau$

$x^{2/3} = \cos^2\tau$ $y^{2/3} = \sin^2\tau$ ∴

$x^{2/3} + y^{2/3} = 1$ and we have eliminated the Parameter τ.

Pg 410



⑦ $Y = \frac{e^x + e^{-x}}{2}$

reflects across Y axis
Looks like a parabola
but it is actually a
catenary.

$Y' = \frac{1}{2}(e^x - e^{-Y})$

$$S = \int_0^b \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2}\,dx \qquad (Y')^2 = \frac{1}{4}(e^x - 2 + e^{-x})$$

$$S = \int_0^b \sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}\,dx =$$

$$S = \int_0^b \sqrt{\left(\frac{1}{2}(e^x + e^{-x})\right)^2}\,dx = \frac{1}{2}\int_0^b e^x + e^{-x}\,dx =$$

$$\frac{1}{2}\,e^x - e^{-x}\Big|_0^b = \frac{1}{2}(e^b - e^{-b}).$$

SKIP. ⑧

$(t + \cos t,\ 2t - \sin t)$ $\quad \frac{ds}{dt}\Big|_{t=1}$

$\dot{x} = 1 - \sin t \qquad \dot{y} = 2 - \cos t$

$(\dot{x})^2 = 1 - 2\sin t + \sin^2 t \qquad (\dot{y})^2 = 4 - 4\cos t + \cos^2 t$

$$\frac{ds}{dt} = \sqrt{\underline{1} - 2\sin t + \underline{\sin^2} t + \underline{4} - 4\cos t + \underline{\cos^2} t}$$

$$= \sqrt{6 - 2\sin t - 4\cos t}\ \Big|_{t=1}$$

$$= \sqrt{6 - 2\sin 1 - 4\cos 1}$$

$$\sqrt{6 - 1.68 - 2.16} = \sqrt{2.156} = 1.468$$

Pg 410

⑨ $(e^{2t}\cos t, t)$ $t=0$

$x = \overset{u}{e^{2t}}\,\overset{v}{\cos t}$

$x' = 2e^{2t}\cos t - e^{2t}\sin t = e^{2t}(2\cos t - \sin t)$

$(x')^2 = e^{4t}(4\cos^2 t - 4\cos t\sin t + \sin^2 t)$

$y = t$

$y' = 1$

$(y')^2 = 1$

$$\left.\frac{ds}{dt}\right|_{t=0} = \sqrt{e^{4t}(4\cos^2 t - 4\cos t\sin t + \sin^2 t) + 1}$$

$$\sqrt{1(4-0+0)+1} = \sqrt{5}$$

⑩ $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ $x=2 \to x=3$

$y' = x - \frac{1}{4x} = \frac{4x^2-1}{4x}$

$(x')^2 = \frac{16x^4-8x^2+1}{16x^2}$

$$S = \int_2^3 \sqrt{\frac{16x^4-8x^2+1}{16x^2}+1}\,dx$$

$$= \int_2^3 \frac{\sqrt{16x^4-8x^2+1+16x^2}}{4x}\,dx = \int_2^3 \frac{\sqrt{16x^4+8x^2+1}}{4x}\,dx =$$

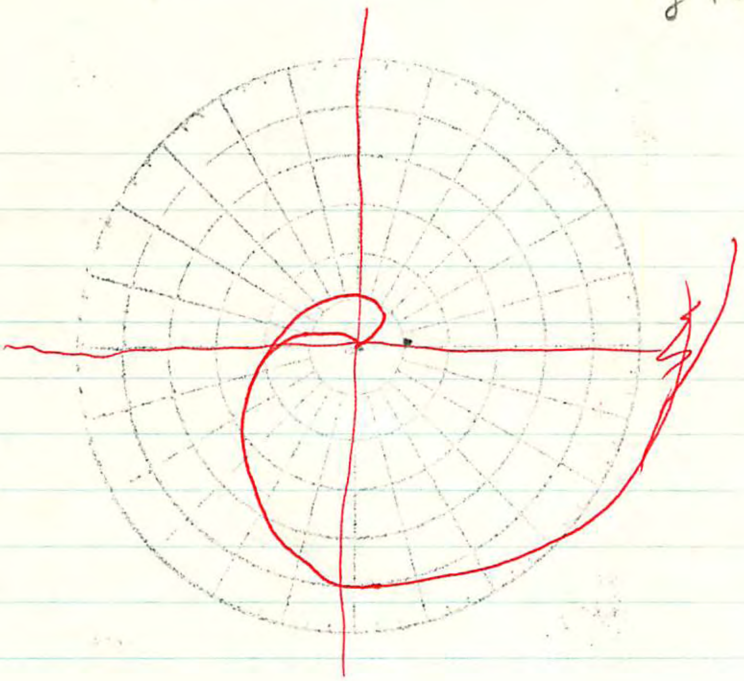
$$\int_2^3 \frac{(4x^2+1)}{4x}\,dx = \int_2^3 x + \frac{1}{4x}\,dx = \left.\frac{1}{2}x^2 + \frac{1}{4}\ln x\right|_2^3$$

$$\frac{9}{2} + \frac{1}{4}\ln 3 - 2 - \frac{1}{4}\ln 2 = \frac{5}{2} + \frac{1}{4}\ln\left(\frac{3}{2}\right)$$

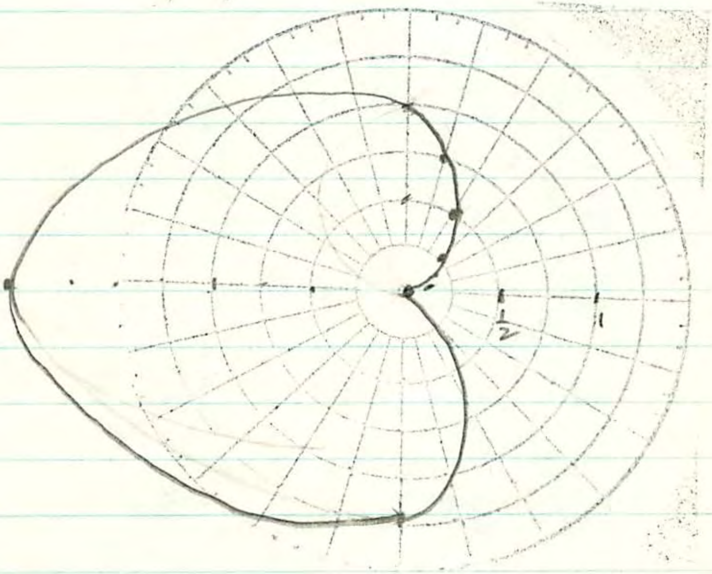
Pg 410

Too hard to draw.

⑪ $r = e^{\theta}$



⑫ $r = 1 - \cos\theta$



θ	R
0	0
15	.034
30	.13
45	.29
60	.50
75	.74
90	1.0
105	

$$2\int_0^{\pi}\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta =$$

$r^2 = 1 - 2\cos\theta + \cos^2\theta$

$r' = \sin\theta$

$(r')^2 = \sin^2\theta$

$$2\int_0^{\pi}\sqrt{2-2\cos\theta} = 2\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\;\frac{\sqrt{1+\cos}}{\sqrt{1+\cos}}\;d\theta$$

$$2\sqrt{2}\int_0^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{1+\cos\theta}}$$

let $u = 1+\cos\theta$

$du = -\sin\theta\,d\theta$

$\theta = 0 \quad u = 2$

$\theta = \pi \quad u = 0$

$$= +2\sqrt{2}\int_0^{2}\frac{du}{u^{\frac{1}{2}}} =$$

$$2\sqrt{2}\cdot 2u^{1/2}\Big|_0^2 = 8$$

Pg 410

⑬ $r = 3\sin\theta \quad [0, \pi]$ $\qquad r^2 = 9\sin^2\theta$

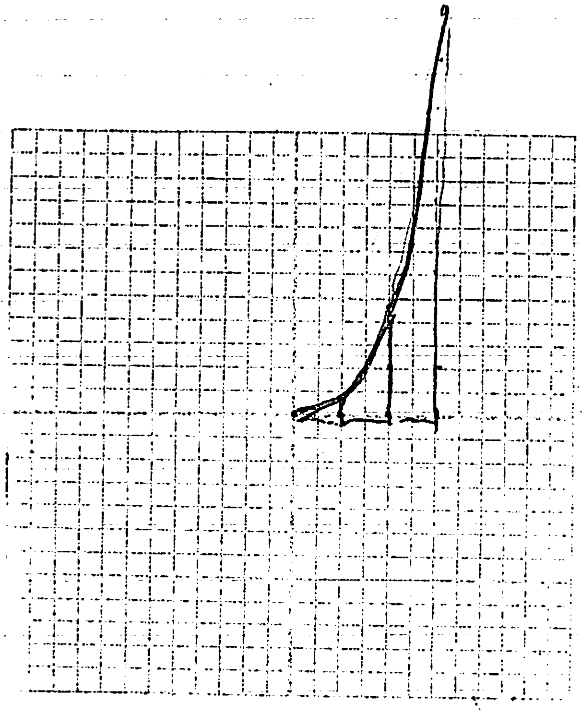
$r' = 3\cos\theta$

$$S = \int_0^{\pi} \sqrt{9\sin^2\theta + 9\cos^2\theta}\, d\theta \qquad (r')^2 = 9\cos^2\theta$$

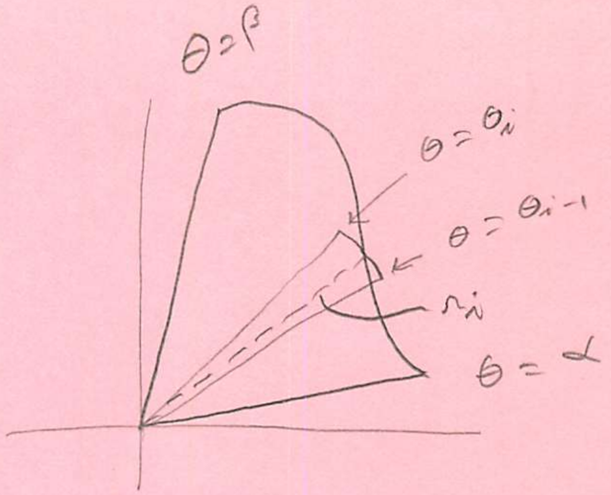
$$S = \int_0^{\pi} 3\, d\theta = 3\theta\Big|_0^{\pi} = 3\pi$$

? ⑭

? ⑮ $y = \frac{1}{3}x^3$



Area in Polar coord. pg 411–415



area of sector of circle is $\frac{r^2\theta}{2}$

we are using sector of circles to approximate the area.

Let ΔA = area of sector bounded by $[\theta_{i-1}, \theta_i]$ and the arc to be $f(\theta)$

For $\Delta\theta$ small then ΔA_i is approximately equal to the circular sector with radius r_i

so $\Delta A_i \approx \frac{1}{2} r_i^2 \Delta\theta$

Now we add all such areas and apply limit as $\Delta\theta \to 0$

$$\sum_{i=\alpha}^{\beta} \frac{1}{2} r_i^2 \Delta\theta$$

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Example area of circle w radius 3.

$$\frac{1}{2}\int_0^{2\pi} 3^2 \, d\theta = \frac{1}{2} 9\theta \Big|_0^{2\pi}$$

$$\frac{9}{\cancel{2}}(\cancel{2}\pi) = 9\pi.$$

Area in Polar
coordinates.

Pg 1

? If $f(\theta)$ is continuous AND nonnegative for $\alpha < \theta < \beta$ and if $0 < \beta - \alpha < 2\pi$ then the area A enclosed by the polar curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$

is
$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

basically we are summing sectors of a circle with varying radius measurements. we take the limit as the changes in the angle goes to zero and end up with the all the sectors which fills the desired area.

remember area for sector of a circle is $\frac{r^2\theta}{2}$

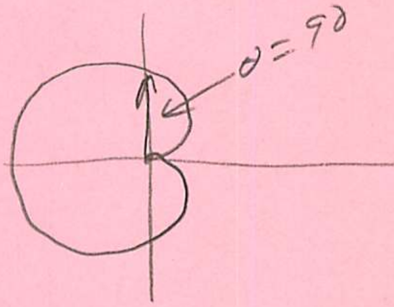
eg circle radius 3 has area 9π
one-quarter of the circle $(\theta = \frac{\pi}{2})$
would give area $\frac{3^2}{2} \cdot \frac{\pi}{2} = \frac{9\pi}{4}$ as we would expect.

The most difficult part of these problems is to determine the limits of integrations

To find those limits.

Draw the picture and see where the "RAY" would start and where it would end while it swept out the desired area. Those will be your limits of integration

eg. cardioid $r = 1 - \cos\theta$ x -4, +4, 1
y -2.6, 2.6, 1

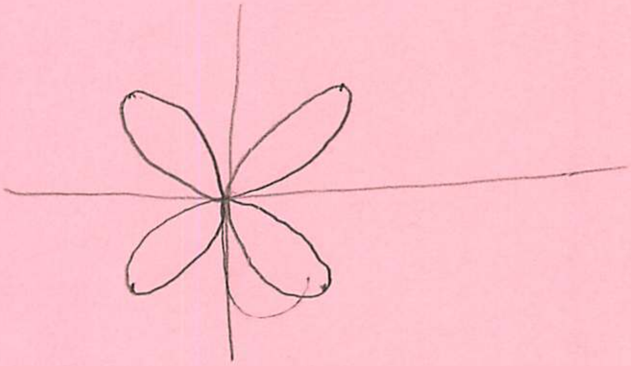


θ starts at 0
θ = 90 straight up
θ = 180 to the left
θ = 270 down
θ = 360 finishes the graph

so $\frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}$

↓

example $r = \sin 2\theta$



Remember the definition says r must be non negative.

$\sin 2\theta$

$2\sin\theta\cos\theta$

$$\begin{array}{ccccccccc} + & 0 & - & 0 & + & 0 & - & 0 \\ 0 & \frac{\pi}{2} & & \pi & & \frac{3\pi}{2} & & 2\pi \end{array}$$

Not sure about this

So we cannot use $\frac{1}{2}\int_0^{2\pi} \sin^2 2\theta \, d\theta$

because

but we CAN See one leaf is $\frac{1}{4}$ of entire figure

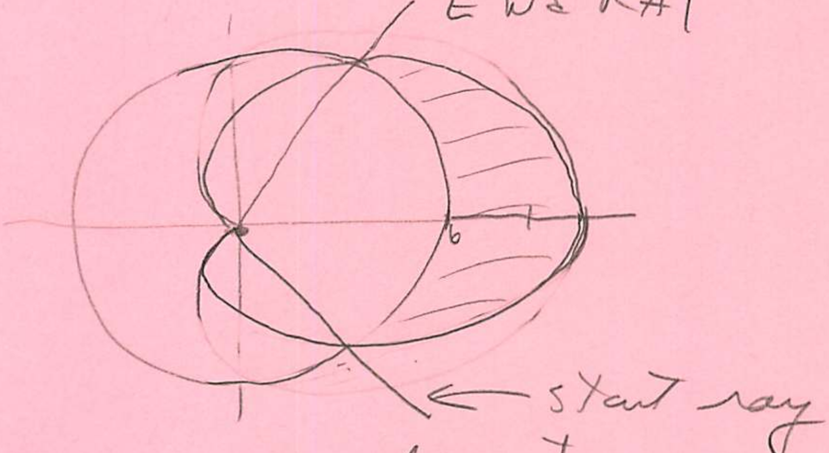
so $4 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \frac{\pi}{2}$

↓

Example: find the area of the region that is inside the cardioid $r = 4+4\cos\theta$ and outside the circle $r=6$

Pg 4

STEP
$\pi/15$



window

θ 0, 2π 2

x −10, 10 1

y −6, 6

we need intersection points.

$$4+4\cos\theta \overset{\text{set}}{=} 6$$

$$4\cos\theta = 2$$

$$\cos\theta = \frac{1}{2}$$

$\theta \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$ as applied to our problem.

$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4+4\cos\theta)^2 - 36 \, d\theta = 18\sqrt{3} - 4\pi$$

18.611

one final note —
sometimes it is difficult or impossible to set polar equations = to each other and solve for "common points"

↓

This is because, in polar graphing, every point has an infinite number of representations. Pg 5

For example

$r_1 = 1 - \cos\theta$ and $r_2 = 1 + \cos\theta$ $\Big\}$

θ $0, 2\pi, \pi/15$
$3, 1, 1$
x $-3, 2, 1$
y $-2, 1$

$1 - \cos\theta \overset{\text{set}}{=} 1 + \cos\theta$

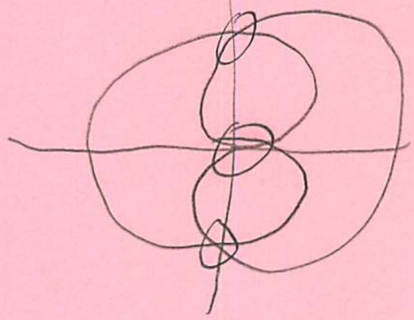
$2\cos\theta = 0$

$\cos\theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. ie $\frac{(2(k+1))\pi}{2}$

but Look at the picture....

Even pictures can be deceptive!! NO!

it has 3 intersection!



ie we missed the one at zero.

~~to avoid problems such as these~~, draw a pict and See how many intersections there should be.

Think.

However...

$r_1(0) = 0$
$r_2(\pi) = 0$

ie both have range value zero but they occur at different times ie are NOT solutions.

.712

EXample 2   Be sure to set mode to POL.

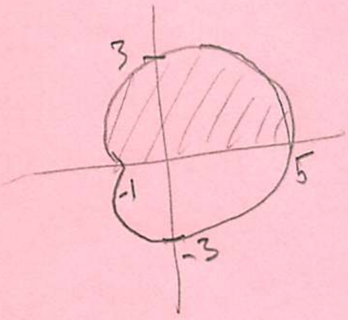
window $\theta_{min} = 0$ $\theta_{max} = 2\pi$

$\theta_{step} = \pi/36 \leftarrow \frac{\pi}{36}$

~~$x \in [-4, 10]$ $y \in [-4, 6]$~~

14 wide

$r = 3 + 2\cos\theta$



$$2 \cdot \frac{1}{2} \int_0^{\pi} (3 + 2\cos\theta)^2 \, d\theta$$

$$\int_0^{\pi} 9 + 12\cos\theta + 4\cos^2\theta \, d\theta$$

$$4\left(\tfrac{1}{2}(1 + \cos 2\theta)\right)$$

$$2 + 2\cos 2\theta$$

$$\int_0^{\pi} 11 + 12\cos\theta + 2\cos 2\theta \, d\theta$$

$$11\theta + 12\sin\theta + \sin 2\theta \Big|_0^{\pi} = 11\pi \approx 34.558$$

-4.5, 7.5

Y

$x\ [-3, 6, 1];\ [-4, 4, 1]$

$\text{fnInt}(r_1{}^2, \theta, 0, \pi)$

$3 + 2\cos\theta \overset{?}{=} 0$

$\cos\theta = -\frac{3}{2}$

Range violation

so $r \neq 0$

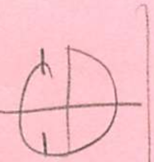
in fact $r > 0$

$\forall\, \theta$.

EXAmple 3

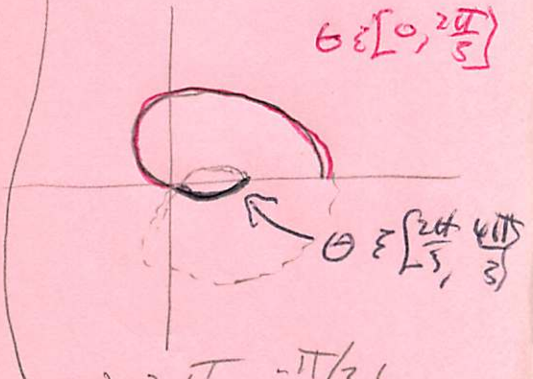
find The area in The Large Loop of The limaçon but outside the inner loop. ~~$x \in [-2, 6]$~~ ~~$y \in [-3, 3]$~~

$r = 1 + 2\cos\theta \overset{set}{=} 0$



← Freedom Rings!

$\cos\theta = -\frac{1}{2} \quad \theta \in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$


$\theta \in [0, \frac{2\pi}{3}]$
$\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$

0 2π π/36

$[-1, 4, 1; -2, 2, 1]$

outer Loop

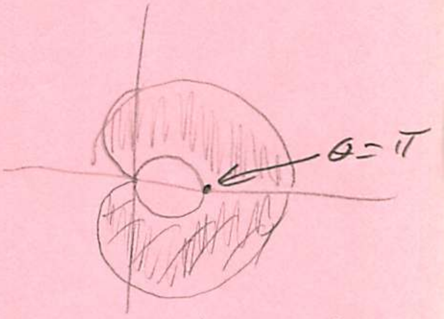
$$\frac{1}{2} \cdot 2 \int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta$$

$$\int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + 4\cos^2\theta \, d\theta$$

$$\int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + 2 + 2\cos 2\theta \, d\theta$$

$$\int_0^{\frac{2\pi}{3}} 3 + 4\cos\theta + 2\cos 2\theta \, d\theta$$

$$3\theta + 4\sin\theta + \sin 2\theta \Big|_0^{\frac{2\pi}{3}} = 2\pi + \frac{3\sqrt{3}}{2}$$



desired area is shaded.

INNER Loop

$$\frac{1}{2} \cdot 2 \int_{\pi}^{\frac{4\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta$$

$$3\theta + 4\sin\theta + \sin 2\theta \Big|_{\pi}^{\frac{4\pi}{3}} = \pi - \frac{3}{2}\sqrt{3}$$

$$2\pi + \frac{3\sqrt{3}}{2} - \left(\pi - \frac{3}{2}\sqrt{3}\right)$$

$$\pi + 3\sqrt{3} \approx 8.3358$$

$$\int_\alpha^\beta \frac{[f(\theta)]^2}{2}\,d\theta \quad \text{or} \quad \int_\alpha^\beta \frac{r^2\,d\theta}{2}$$ Pg 414

① $r = 2\theta \quad \alpha = 0 \quad \beta = \frac{\pi}{2}$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} 4\theta^2\,d\theta = \frac{2\theta^3}{3}\Big|_0^{\frac{\pi}{2}} = \frac{2}{3}\,\frac{\pi^3}{8} = \frac{\pi^3}{12} \approx 2.584$$

② $r = \sqrt{\theta} \quad \alpha = 0 \quad \beta = \pi$

$$\frac{1}{2}\int_0^{\pi} \theta\,d\theta = \frac{\theta^2}{4}\Big|_0^{\pi} = \frac{\pi^2}{4} \approx 2.467$$

③ $r = \frac{1}{1+\theta} \quad \alpha = \frac{\pi}{4} \quad \beta = \frac{\pi}{2}$

$$r^2 = \frac{1}{(1+\theta)^2}$$

$$\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{(1+\theta)^2} = -\frac{1}{2}(1+\theta)^{-1}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{2}\left(1+\frac{\pi}{2}\right)^{-1} + \frac{1}{2}\left(1+\frac{\pi}{4}\right)^{-1}$$

$$= -\frac{1}{2}\left(\frac{2}{2+\pi} - \frac{4}{4+\pi}\right) = -\frac{1}{2}\left(\frac{8+2\pi-8-4\pi}{(2+\pi)(4+\pi)}\right) = \frac{\pi}{(2+\pi)(4+\pi)}$$

.086

④ $r = \sqrt{\sin\theta} \quad \alpha = 0 \quad \beta = \frac{\pi}{2}$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta = -\frac{1}{2}\cos\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{2} = .5$$

~~Book Ans~~ Wrong.

⑤ $r = \tan\theta \quad \alpha = 0 \quad \beta = \frac{\pi}{4}$

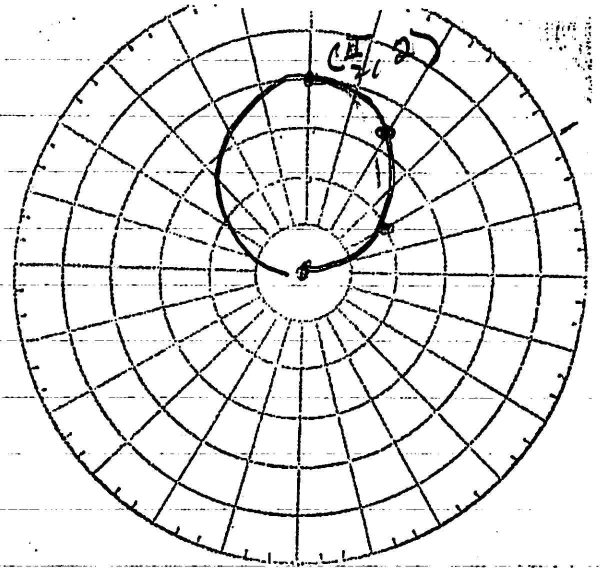
$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \tan^2\theta\,d\theta = \frac{1}{2}(\tan\theta - \theta)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2} - \frac{\pi}{8} = .107$$

Pg 414

⑥ $r = 2\sin\theta$

θ	R
0	0
$\frac{\pi}{6}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	$\sqrt{3}$
$\vdots$	$\vdots$

full curve is swept out fm $0 \to \pi$



$$\frac{1}{2}\int_0^{\pi} 4\sin^2\theta\, d\theta =$$

$$\frac{4}{2}\cdot\frac{1}{2}\int_0^{\pi} 1-\cos 2\theta\, d\theta = \theta + \frac{1}{2}\sin 2\theta\Big|_0^{\pi} = \pi$$

7 Too hard to graph

$\frac{1}{2}\int_0^{\pi} (3+3\sin\theta)^2 \, d\theta$ −

$\int \frac{1}{2}(1-\cos 2\theta)$

$\int \sin^2 =$

$\frac{9}{2}\int_0^{\pi} (1+\sin\theta)^2 \, d\theta$

$3\int_0^{\pi} 1 + 2\sin\theta + \sin^2\theta \, d\theta$

~~$\frac{1}{2}\left\{ 3\theta - 6\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right|_0^{\pi}$~~

~~$\frac{1}{2}\left\{ 3\pi + \overset{3}{\cancel{6}} + \frac{\pi}{2} \right]$~~

$\frac{27\pi}{4} + 3$

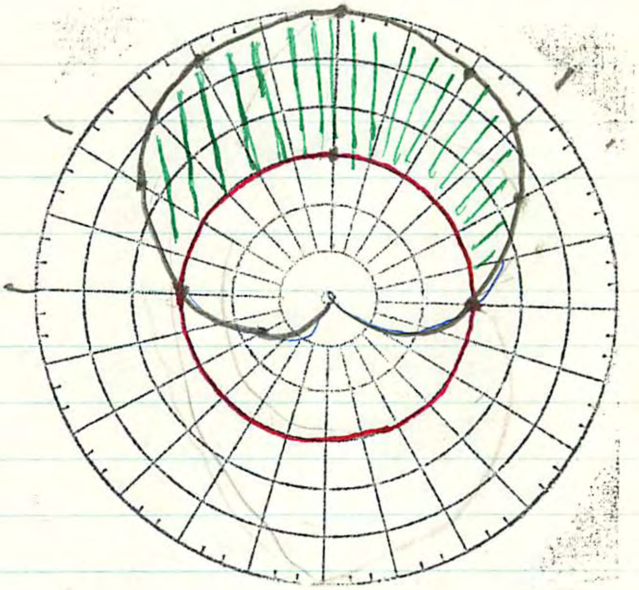
$\left(\frac{9\theta}{4} - \frac{9\sin 2\theta}{4} - 9\cos\theta + \frac{9\theta}{2} - \frac{9\pi}{2} \right) \Big|$

$18 + \frac{9\pi}{4}$

Pg 414

8 $r = 3 + 3\sin\theta$

$r = 3$



θ	$3\sin\theta$	$3+3\sin\theta$
0	0	3
$\frac{\pi}{6}$	$\frac{3}{2}$	$4\frac{1}{2}$
$\frac{\pi}{4}$	2.12	5.12
$\frac{\pi}{3}$	2.26	5.26
$\frac{\pi}{2}$	3	6
$\frac{2\pi}{3}$	2.6	5.6
$\frac{3\pi}{4}$	2.12	5.12
$\frac{5\pi}{6}$	1.5	4.5
π	0	3
$\frac{7\pi}{6}$	$-\frac{3}{2}$	1.5

area of interest of circle is Top 1/2

Curve finishes at 2π

$\therefore$ area of Cardiod $\frac{1}{2}\int_0^{2\pi} (3+3\sin\theta)^2 \, d\theta$

our area of interest $\frac{1}{2}\int_0^{\pi}\left((3+3\sin\theta)^2 - 3^2\right) d\theta =$

$$\frac{9}{2}\int_0^{\pi} 2\sin\theta + \sin^2\theta \, d\theta = 9\int_0^{\pi} 2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$\frac{9}{2}\left[-2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi} = \frac{9}{2}\left[+2 + \frac{\pi}{2} + 2\right] =$$

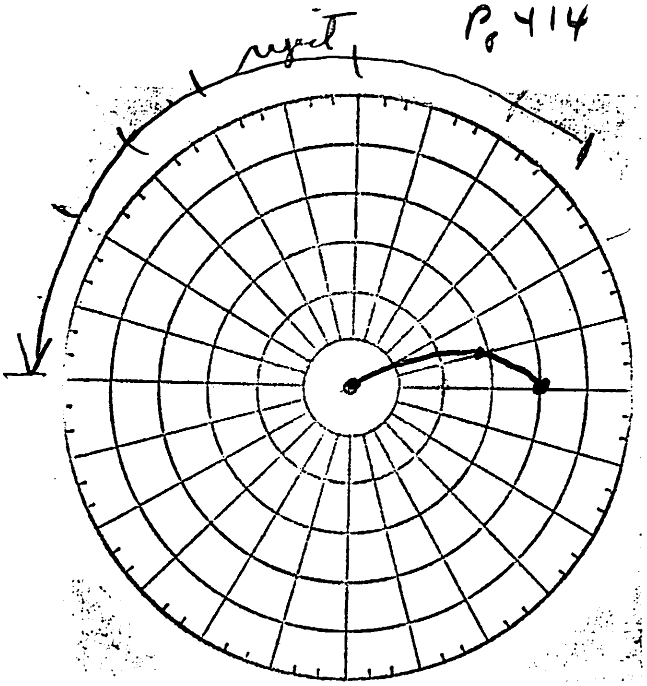
$18 + \frac{9\pi}{4}$ ⇐ given as answer to #9 in back of book.

Pg 414

Skip

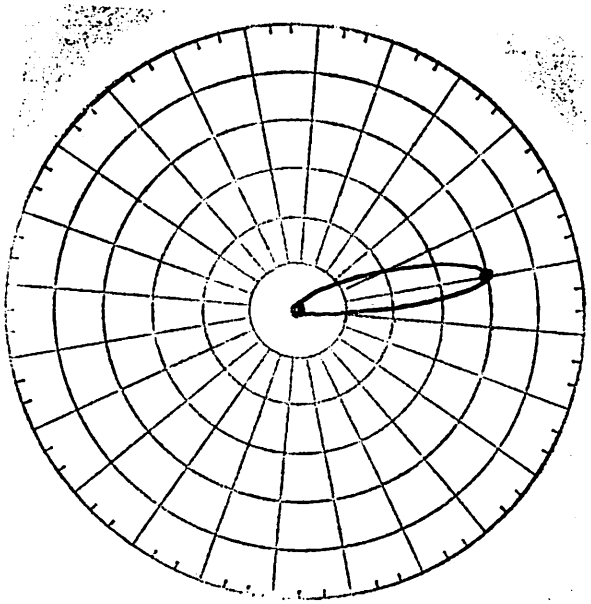
⑨ $r = \sqrt{\cos 2\theta}$

θ	$\cos 2\theta$	$\sqrt{\cos 2\theta}$
0	1	1
$\frac{\pi}{6}$	.5	.707
$\frac{\pi}{4}$	.0	0
↓	reject	
	0	0
$\frac{2\pi}{6}$		



⑩ $r = \sin 3\theta$

θ	$\sin 3\theta$
0	0
$\frac{\pi}{6}$	1 completes one leaf.
$\frac{\pi}{3}$	0

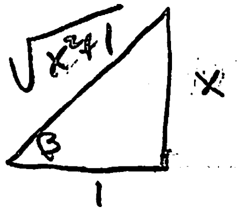


$$\frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2 3\theta \, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} 1 - \cos 6\theta \, d\theta =$$

$$\frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_0^{\frac{\pi}{3}} = \frac{1}{4}\left(\frac{\pi}{3}\right) = \frac{\pi}{12}$$

Pg 414

14 $\int_0^\beta \frac{1}{2}\sec^2\theta\, d\theta$ =



a) $r = \sqrt{x^2+1}$ ie length of r is a fctn of length of x r rotates from 0 to β.

$\frac{1}{2}\int_0^\beta x^2+1\, d\theta$ = Notice $\sec\beta = \sqrt{x^2+1}$ ∴ $x^2+1 = \sec^2\beta$ so

$\frac{1}{2}\int_0^\beta \sec^2\theta\, d\theta$

b) $\frac{1}{2}\cdot 1\cdot x = \frac{x}{2}$ = Area of Δ.

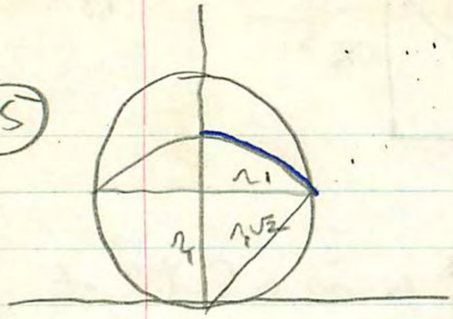
but $x = \tan\beta$ ∴ area of Δ is $\frac{1}{2}\tan\beta$

$d\left(\frac{1}{2}\tan\beta\right) = d\left(\int_0^\beta \frac{1}{2}\sec^2\theta\, d\theta\right)$ use 2nd FTC Pg 290

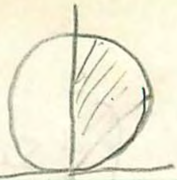
$\Downarrow$

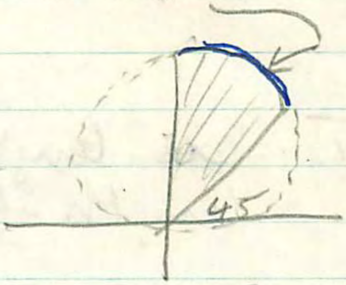
$\frac{1}{2}\sec^2\theta = \frac{1}{2}\sec^2\theta\Big|_{\theta=\beta} = \frac{1}{2}\sec^2\beta$

⑮

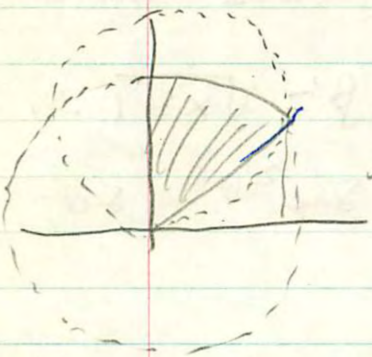


$r = 2r_1 \sin\theta \longrightarrow$

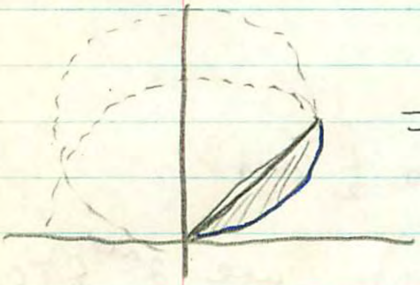




$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2r_1 \sin\theta)^2 \, d\theta$



$r = \sqrt{2} r_1 = \int_{\frac{\pi}{4}}^{\pi} (\sqrt{2})^2 \, d\theta$



$= \int_{0}^{\frac{\pi}{4}} 2r_1 \sin\theta \, d\theta$

$4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin^2\theta}{2} \, d\theta - 2\int_{\frac{\pi}{4}}^{\pi} \frac{1}{2} \, d\theta$ w/ rs $= \frac{r^2}{2}$

$\cancel{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{\cancel{2}} \, d\theta - \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

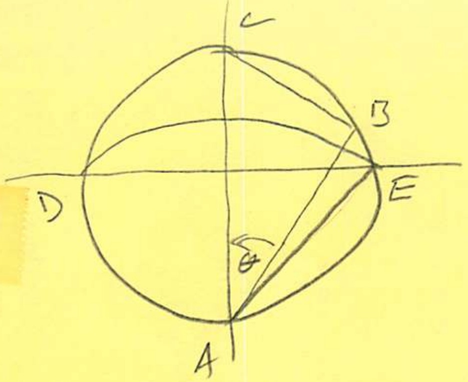
$\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$

$\theta - \frac{\sin 2\theta}{2} - \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{\sin 2\theta}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$-0 + \frac{1}{2} = \frac{1}{2}$ ✓

Pg 414

#15



Symmetric

$AE = r\sqrt{2}$

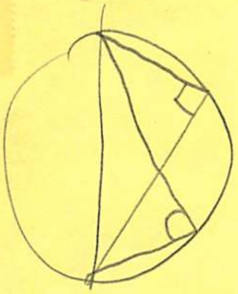
$AB = \underline{2r\cos\theta}$ How? see lower left corner.

Note! when B is at E
$\theta = \frac{\pi}{4}$
when B is at C
$\theta = 0$
area is symmetric
so double integral

$$2\int_0^{\frac{\pi}{4}} \frac{4r^2\cos^2\theta}{2} - \frac{2r^2}{2}\,d\theta$$

$$2r^2\int_0^{\frac{\pi}{4}} 2\cos^2\theta - 1\,d\theta$$

$$2r^2\int_0^{\frac{\pi}{4}} \cancel{2}\left[\tfrac{1}{\cancel{2}}(1+\cos 2\theta)\right] - 1\,d\theta$$

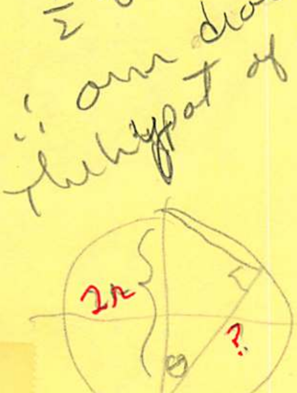


$$2r^2\int_0^{\frac{\pi}{4}} \cos 2\theta\,d\theta$$

INscribed ∡ is ½ of central ∡
∴ any diameter is the hypot of rt triangle.

$$2r^2\left[\tfrac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{4}}\right.$$

$$2r^2\left[\tfrac{1}{2}\sin\tfrac{\pi}{2} - 0\right]$$



$$2r^2\left[\tfrac{1}{2}\right]$$

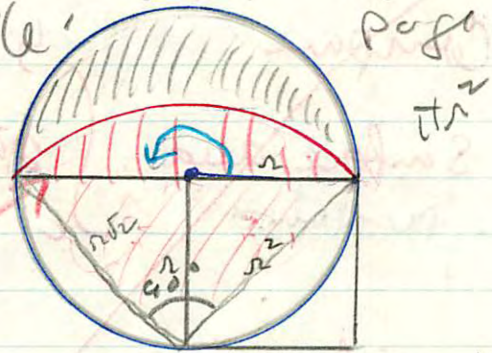
$\frac{adj}{hyp} = \frac{?}{2r} = \cos\theta$
$? = 2r\cos\theta$.

r^2 which is the area of the square.

This method had $\frac{d}{\pi r^2}$ Pg 414

NO STYLE. See previous page.

⑮ WTS Shaded area $= r^2$



upper arc Generates Semi circle

$$\frac{1}{2}\int_0^{\pi} r^2\,d\theta$$

lower arc created by ray of length $r\sqrt{2}$

Sector $- 2\cdot\frac{1}{2}r^2 =$ Sector $- r^2$
↑ the two Δs

actually its $\frac{1}{4}$ circle of radius $r\sqrt{2}$ so area is $\frac{2\pi r^2}{4} = \frac{\pi r^2}{2}$

area of Sector $= \frac{\theta r^2}{2}$ (Pg 86) here radius is $r\sqrt{2}$ and $\theta = \frac{\pi}{2}$

$\frac{\pi}{2}\cdot 2r^2$

$$\frac{\pi r^2}{2} - r^2 = \frac{r^2}{2}(\pi - 2)$$

putting it all together.

upper Semi circle →

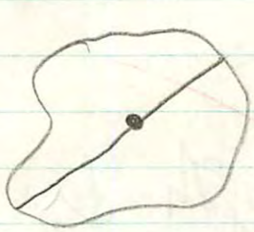
$$\frac{1}{2}\int_0^{\pi} r^2\,d\theta - \frac{\pi r^2}{2} + r^2$$

$$\frac{r^2\theta}{2}\Big|_0^{\pi} - \frac{\pi r^2}{2} + r^2$$

$$\frac{r^2\pi}{2} - \frac{\pi}{2}r^2 + r^2 = r^2.$$

two Abstract This Time I think.. or ever.

⑯



Yes, Let P be the Pole and let the boundary of R be $r = f(\alpha)$.
assume $\alpha = \theta$ START Then as θ moves to $\pi + \theta$ it will sweep out ½ of area. ie

$$\frac{A}{2} = \int_{\theta}^{\theta+\pi} \frac{1}{2}\left[f(\alpha)\right]^2 d\alpha$$

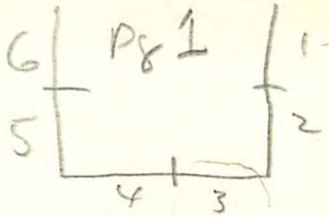
diff each side WRT α using 2nd FTC.

$$0 = \frac{1}{2}\left[f(\theta+\pi)\right]^2 - \frac{1}{2}\left[f(\theta)\right]^2 =$$

$$\frac{1}{2}\left[f(\theta+\pi)\right]^2 = \frac{1}{2}\left[f(\theta)\right]^2 \Rightarrow f(\theta+\pi) = f(\theta)$$

So the chord determined by θ has as its centre the Pole ie P is midpt.

Tell students to WATCH only
DON'T TAKE NOTES until the end.
There's too much "Jumping around" to take notes efficiently



On Board #3

Say " Given a cone with height h and radius r, find the surface area"

" this could be restated ; bounded in the 1ST quadrant :
Given a line $y = -\frac{r}{h}x + r$, Revolve it around the x axis, ~~the line is bounded in the 1ST quadrant~~ "

Some people say that it is similar to the "shell method" of finding area.

What we want to do is sum up each of the lines formed when we rotate our given line. The formula we need is :

Shown on Board #3

$$\int_a^b 2\pi \; y \; \underbrace{\sqrt{1+(y')^2}} \; dx$$

this Revolves our line about this axis

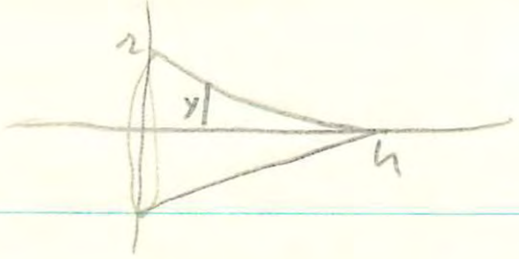
our Typical radius of Revolution (in terms of)

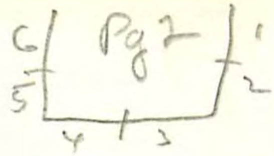
this is the length of the curve being revolved. "

NOTE the "Length" requires the integral too. ie $\int ds$ is the length.

IT IS not enough to "Just multiply the length of the line times the integral $2\pi \int y \, dx$

Go To Board #1





On Board #1

" For our problem, the limits are from 0 to h. a typical radius of revolution is $y = -\frac{rx}{h} + r$, so $y' = -\frac{r}{h}$ using the formula on the front board"

So we get

$$\int_0^h 2\pi\left(-\frac{rx}{h}+r\right)\sqrt{1+\frac{r^2}{h^2}}\,dx =$$

$$2\pi\int_0^h \frac{-rx+rh}{h}\sqrt{\frac{h^2+r^2}{h^2}}\,dx = \frac{2\pi}{h}\int_0^h(-rx+rh)\frac{\sqrt{h^2+r^2}}{h}dx$$

$$\frac{2\pi\sqrt{h^2+r^2}}{h^2}\int_0^h -rx+rh\,dx =$$

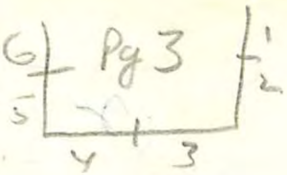
$$\frac{2\pi\sqrt{h^2+r^2}}{h^2}\left(-\frac{rx^2}{2}+rhx\right)\Big|_0^h =$$

$$\frac{2\pi\sqrt{h^2+r^2}}{h^2}\left(-\frac{rh^2}{2}+rh^2\right) = \left(\frac{\cancel{2}\pi\sqrt{h^2+r^2}}{\cancel{h^2}}\right)\left(\frac{r\cancel{h^2}}{\cancel{2}}\right) =$$

$$\pi r\sqrt{h^2+r^2}$$

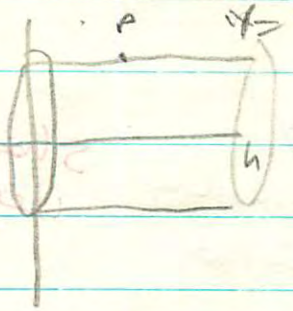
Go to Board #5

Other people think we should just sum circumferences "instead of using a complicated formula."



On Board #5

consider the line $y = 2$ "lets use summation of circumferences"



"If we revolve a point around the x axis and move from 0 to h we would sweep out the surface of this cylinder consider $2\pi r$ as circumference of our typical circle

this: $2\pi\int_0^h y\,dx$ seems to describe the problem.

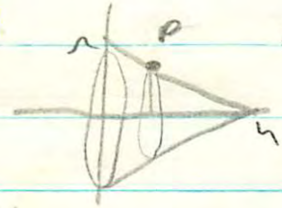
$$2\pi\int_0^h 2\,dx = 2\pi(2x)\Big|_0^h = 4\pi h$$

ie $\underbrace{(2\pi)(r)}(h)$

(circum of circle)(height) seems to say "this method is OK" But, of course, it is Not!

On Board #6

consider



$y = -\frac{r}{h}x + r$ "lets sum all circumferences here too."

ie $2\pi r$ = circum of each circle

$$2\pi\int_0^h y\,dx = 2\pi\int_0^h -\frac{r}{h}x + r\,dx =$$

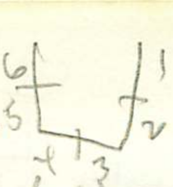
$$2\pi\left(-\frac{rx^2}{2h} + rx\right)\Big|_0^h = 2\pi\left(-\frac{rh^{\cancel{2}}}{2\cancel{h}} + rh\right) = 2\pi\left(\frac{rh}{2}\right) = \pi rh$$

"Doesn't it seem strange that we get two such different representations for the same surface area?"

Go to board #2

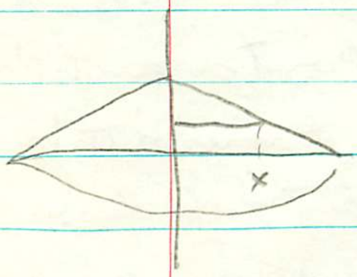
Pg 14'

on Board 2



(This version I try to get away using a "blind" substitution)

Let's rotate about y axis instead of x axis



$$2\pi\int_0^r x\sqrt{1+(y')^2}\,dy$$

Noted error

Should have been $\frac{dx}{dy}$

$$2\pi\int_0^r\left(-\frac{h}{r}y+h\right)\sqrt{1+\frac{r^2}{h^2}}\,dy$$

$$2\pi\int_0^r\frac{-hy+rh}{r}\,\frac{\sqrt{h^2+r^2}}{h}\,dy$$

$$\frac{2\pi}{r}\int_0^r(-y+r)\sqrt{h^2+r^2}\,dy$$

$$\frac{2\pi}{r}\sqrt{h^2+r^2}\int_0^r -y+r\,dy$$

$$\frac{2\pi}{r}\sqrt{h^2+r^2}\left(-\frac{y^2}{2}+ry\right)\Bigg|_0^r$$

$$\frac{2\pi}{r}\sqrt{h^2+r^2}\left(-\frac{r^2}{2}+r^2\right)$$

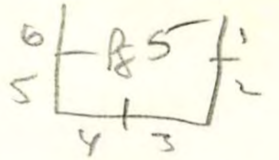
$$\frac{\cancel{2}\pi}{\cancel{r}}\sqrt{h^2+r^2}\left(\frac{r^{\cancel{2}}}{\cancel{2}}\right)$$

$$\pi r\sqrt{h^2+r^2}$$

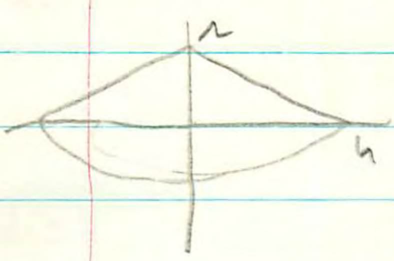
But This is The Same As Board #1 Answer!!

"Same Line is Rotated through 360° so Area swept out is the Same" What is wrong with this statement?

ALSO on Board 2



ALSO
on Board #2 "lets rotate about y axis instead of x."



$y = -\frac{r}{h}x + r$

$$2\pi\int_0^r x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

↑ this is correct in this version

$hy = rx + rh$

$rx = -hy + rh$

$x = -\frac{h}{r}y + h$

$\frac{dx}{dy} = -\frac{h}{r}$

$$2\pi\int_0^r\left(-\frac{h}{r}y + h\right)\sqrt{1+\frac{h^2}{r^2}}\,dy$$

$$2\pi\int_0^r\frac{-hy+rh}{r}\sqrt{\frac{r^2+h^2}{r^2}}\,dy$$

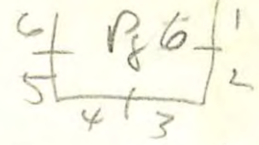
$$\frac{2\pi\sqrt{r^2+h^2}}{r^2}\int_0^r -hy+rh\,dy$$

$$\frac{2\pi\sqrt{r^2+h^2}}{r^2}\left(-\frac{hy^2}{2}+rhy\right)\Big|_0^r = \frac{2\pi\sqrt{r^2+h^2}}{r^2}\left(-\frac{hr^2}{2}+hr^2\right)$$

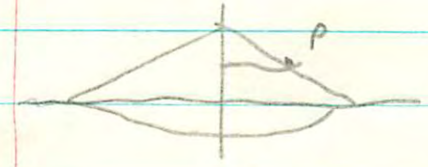
$$\frac{2\pi\sqrt{r^2+h^2}}{r^2}\left(hr^2\right) = 2\pi h\sqrt{r^2+h^2}$$

Go to Board 4

add circumferences again



on Board #4 "lets rotate about y axis in terms of y"



$y = -\frac{r}{h}x + r$

so

$2\pi\int_0^r x\,dy$ $x = -\frac{h}{r}y + h$

$$2\pi\int_0^r\left(-\frac{h}{r}y+h\right)dy$$

$$2\pi\left(-\frac{hy^2}{2r}+hy\right)\Big|_0^r = 2\pi\left(-\frac{hr^2}{2r}+hr\right) =$$

$$2\pi\left(\frac{-hr^2+2hr^2}{2r}\right) = \cancel{2}\pi\left(\frac{hr^{\cancel{2}}}{\cancel{2}\cancel{r}}\right) = \pi r h.$$

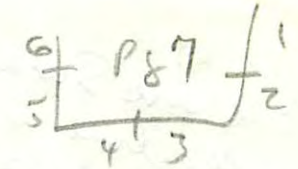
Notice that is is the same as value on Board #6 (Bottom of pg 3)

"well, we rotated the same line through the same 360°. what do you expect?"

"ok, ok, ... there is a problem ....

what is the problem?

Adding circumferences of circles



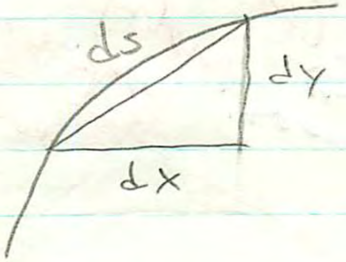
our use of $\int 2\pi y\,dx$ is improper.

IT should be written $\int 2\pi y\,ds$ (Pg 417)

$ds = \frac{ds}{dx} \cdot dx$ because we are moving along the line not along some axis

thus the above becomes $\int 2\pi y \frac{ds}{dx} \cdot dx$ ie "per change in the line" rather than "per change in an axis"

in our approximations (Pg 409 implied on)



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

So, Now, we get $y = -\frac{r}{h}x + r$ $\frac{dy}{dx} = -\frac{r}{h}$

on Board #1 we did it properly:
(Pg 2)

$$2\pi \int_0^h \left(-\frac{r}{h}x + r\right)\sqrt{1 + \left(\frac{r^2}{h^2}\right)}\,dx$$

$$2\pi r \int \left(-\frac{x}{h} + 1\right)\frac{\sqrt{h^2 + r^2}}{h}\,dx$$

$$\frac{2\pi r\sqrt{h^2+r^2}}{h^2}\int_0^h -x + h\,dx$$

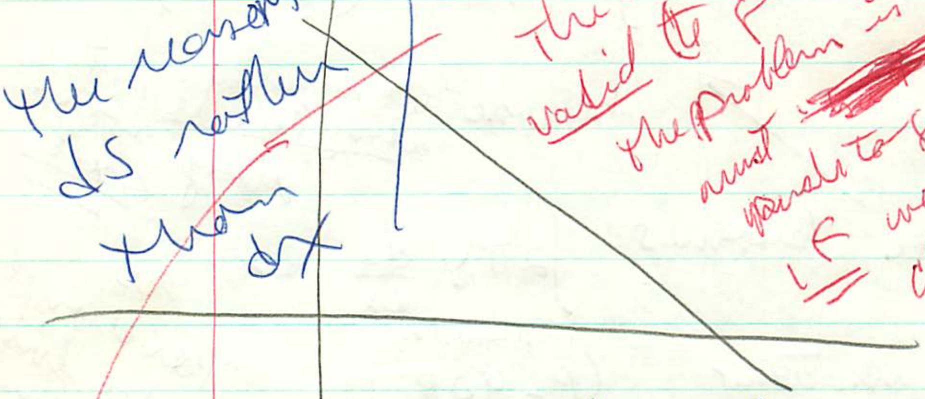
$$\frac{2\pi r\sqrt{h^2+r^2}}{h^2}\left[-\frac{x^2}{2} + hx\right|_0^h$$

$$\frac{2\pi r\sqrt{h^2+r^2}}{h^2}\left[\frac{h^2}{2}\right] = \pi r\sqrt{h^2+r^2}$$

we are Summing up the points to make a line using dS because we are "following the line" instead of changing "per change in x".

the reason for dS rather than dX ↑

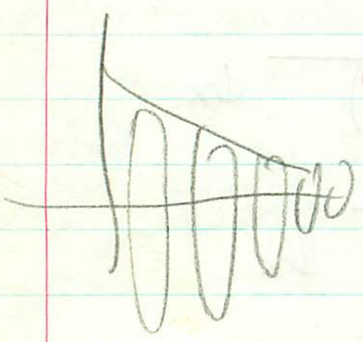
this is really the key. the idea of adding circumferences is valid to get surface area) the problem is that circumferences must [illegible] pushed together as they do IF we move them per the change in the line as in the picture. contrast to what (as con pushing together as happens in this method.



"following the line" means we changed from (X, Y) to (X+ΔX, Y+ΔY) then from (X+ΔX, Y+ΔY) to the Next point and on and on.

$2\pi\int y\,dx$
↑

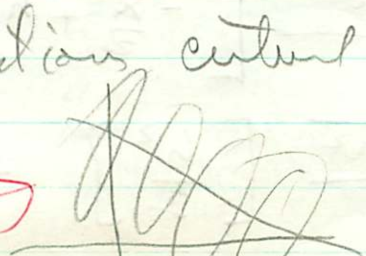
X means cross sections are centered on X axis



We want to move along X but formula to do that requires S

$2\pi\int y\,ds$
↑

means cross sections centered on line S



Why then does our former "ERROR" method seem to work when we use a cylinder?

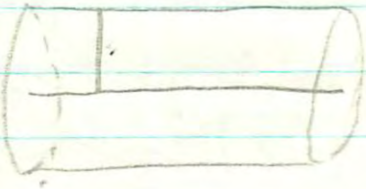
$$\Delta S = \overset{2\pi Y}{2\pi Y}\,\overset{dx}{\Delta x} \leftarrow \text{from "ERROR" method}$$

$$\Delta S = \overset{2\pi y}{2\pi Y}\,\overset{dS}{\sqrt{(\Delta x)^2+(\Delta Y)^2}} \leftarrow \text{from "PROPER" method.}$$

$$\frac{2\pi Y\sqrt{(\Delta x)^2+(\Delta Y)^2}}{2\pi Y\Delta x} \qquad \frac{\text{Proper}}{\text{ERROR}}$$

$$\frac{\sqrt{(\Delta x)^2+(\Delta Y)^2}}{\Delta x} = \sqrt{1+\left(\frac{\Delta Y}{\Delta x}\right)^2}$$

Thus if the ratio is 1 then all would be well, ie if $\frac{dY}{dx}$ is zero.



Y is the radius. in a cylinder since there is no change in Y as X moves from 0 to h $\frac{dY}{dx}$ is zero, because Y=r

and $\sqrt{1+\left(\frac{\Delta Y}{\Delta x}\right)^2}$ is $\sqrt{1+0} = \sqrt{1} = 1$

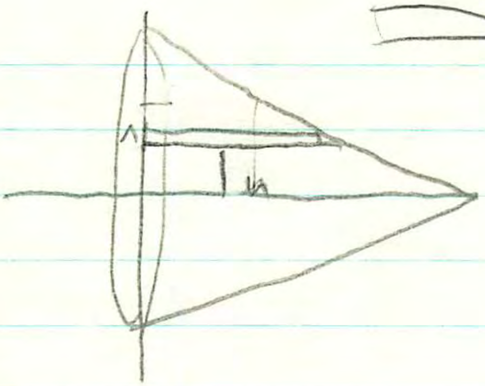
thus "it is never missed" leading us to the wrong conclusion.

Now students can write notes from stuff on the boards.

To contrast our surface area problems

Volume using Shells

Pg 9



$y = -\frac{r}{h}x + r$

$y - r = -\frac{r}{h}x$

$r - y = \frac{r}{h}x$

$h - \frac{hy}{r} = x$

$2\pi\int_0^r y\ x\,dy$

radius

length

Sample moves Vertically
∴ y is changing
Thus, the sample is
NOT moving "on some
angle"

$2\pi\int_0^r yh - \frac{hy^2}{r}\,dy$

$2\pi h\int_0^r y - \frac{y^2}{r}\,dy$

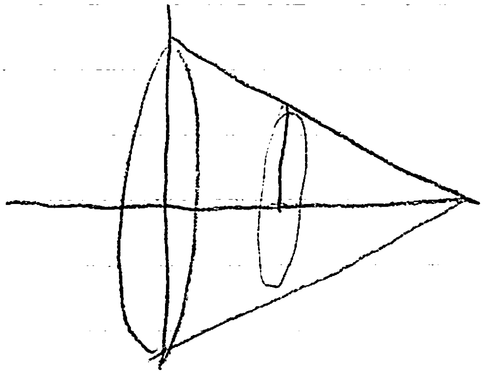
$2\pi h\left[\frac{y^2}{2} - \frac{y^3}{3r}\right]\Big|_0^r$

$2\pi h\left[\frac{r^2}{2} - \frac{r^2}{3}\right]$

$2\pi h\left[\frac{r^2}{6}\right]$

$\frac{\pi r^2 h}{3}$

Volume using ~~circles~~ disks



$Y = -\frac{r}{h} x + r$

$\pi \int r^2 \, dx$ ← circles move horizontally

again... Not drawing on same slant.

$$\pi \int_0^h \left(-\frac{r}{h} x + r\right)^2 dx$$

$$\pi \int_0^h \frac{r^2 x^2}{h^2} - \frac{2 r^2 x}{h} + r^2 \, dx$$

$$\pi \left[\frac{r^2 x^3}{3 h^2} - \frac{r^2 x^2}{h} + r^2 x\right]\Bigg|_0^h$$

$$\pi \left[\frac{r^2 h}{3} - r^2 h + r^2 h\right] = \frac{\pi r^2 h}{3}$$

Pg 420

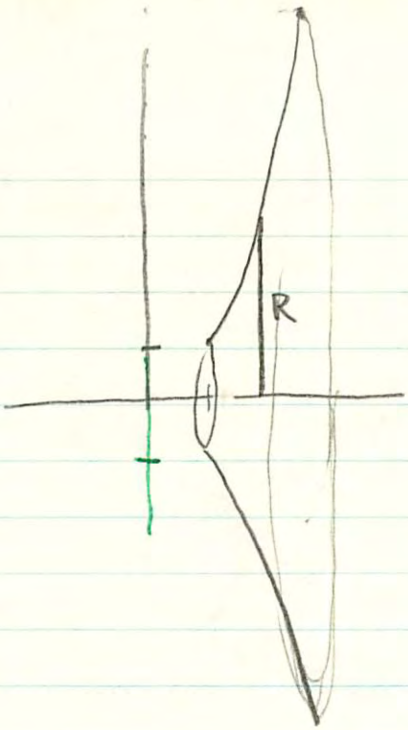
$y' = 3x^2$

① $y = x^3$ $[1, 2]$

$$2\pi\int_1^2 x^3\sqrt{1+9x^4}\,dx$$

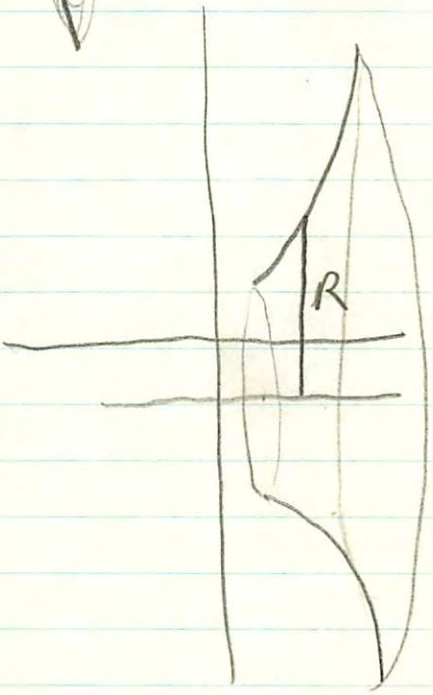
let $u = 1+9x^4$

$du = 36x^3$ etc.



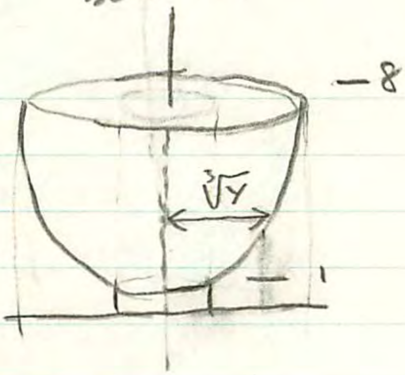
② $y = x^3$ $[1, 2]$ about $y = -1$

$$2\pi\int_1^2 (x^3+1)\sqrt{1+9x^4}\,dx$$



Parameter Y $\quad x = Y^{\frac{1}{3}} \quad dx = \frac{1}{3}Y^{-2/3}\,dY$

bowl w/ flat bottom

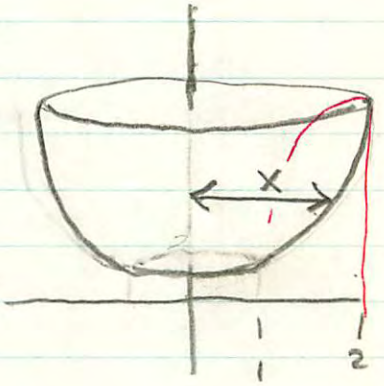


③ $Y = x^3 \quad [1,2]$ about y axis

$$2\pi\int_1^8 \sqrt[3]{Y}\,\sqrt{1+\frac{1}{9Y^{4/3}}}\,dy$$

$$2\pi\int_1^8 \sqrt[3]{Y}\,\frac{\sqrt{9Y^{4/3}+1}}{3Y^{2/3}}\,dy$$

$$\frac{2\pi}{3}\int_1^8 \frac{\sqrt{9Y^{4/3}+1}}{Y^{1/3}}\,dy$$



④ $y = x^3 \quad [1,2]$ Parameter x

$y' = 3x^2$

$3x^2$

$$2\pi\int_1^2 x\sqrt{1+9x^4}\,dx$$

⑤ $r = \sin 2\theta \quad \theta \in [0, \frac{\pi}{4}]$ about polar axis Parameter θ.

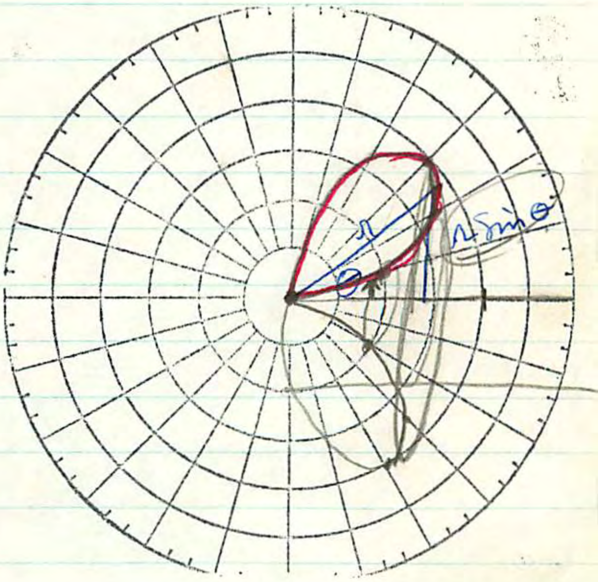
$r' = 2\cos 2\theta$

$$2\pi\int_0^{\frac{\pi}{2}} \sin 2\theta \sin\theta\sqrt{\sin^2 2\theta + 4\cos^2 2\theta}\,d\theta$$

$$r\sin\theta\sqrt{r^2+(r')^2}\,d\theta.$$

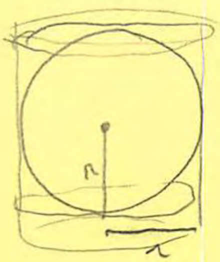
remember $Y = r\sin\theta$ ↑ the vertical length which is the radius

$$\int \sqrt{r^2+r^2}\,d\theta$$



Pg 420-421

⑥



Volume of Sphere $\frac{4}{3}\pi r^3$

Vol. of cylinder $\pi r^2 h \rightarrow 2\pi r^3$

$\pi r^2(2r)$

a) $$\frac{\frac{4}{3}\pi r^3}{2\pi r^3} = \frac{2}{3}$$

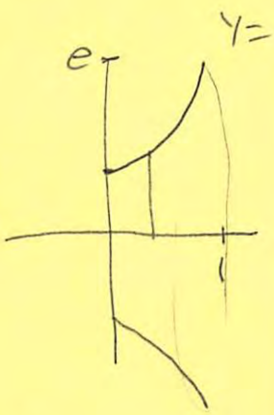
b) Surface area of Sphere $4\pi r^2$

Surface area of Cylinder $(2\pi r)2r = 4\pi r^2$

(ignoring top and bottom)

They are The same

⑦



$$2\pi\int_0^1 y\sqrt{1+e^{2x}}\,dx$$

$$2\pi\int_0^1 e^x\sqrt{1+e^{2x}}\,dx$$

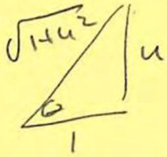
$$2\pi\int_1^e \sqrt{1+u^2}\,du$$

let $u = e^x$

$du = e^x\,dx$

$x=0 \rightarrow u=1$

$x=1 \rightarrow u=e$



let $\tan\theta = u$

$\sec^2\theta\,d\theta = du$

$u=1 \quad \theta = \frac{\pi}{4}$

$u=e \quad \theta = \tan^{-1}e$

$\sec\tan^{-1} = \frac{1}{\cos\tan^{-1}u}$

$\sqrt{1+u^2}$

$$2\pi\int \sec\theta\sec^2\theta\,d\theta$$

$$2\pi\int_{\frac{\pi}{4}}^{\tan^{-1}e} \sec^3\theta\,d\theta$$

$$2\pi\cdot\frac{1}{2}\left[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right]\Big|_{\frac{\pi}{4}}^{\tan^{-1}e}$$

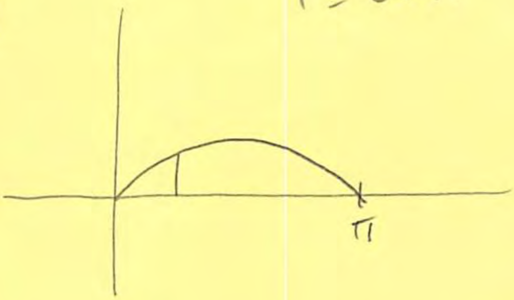
$$\pi\left[\sec\tan^{-1}e\,(e) - \ln|\sec\tan^{-1}e + e| - \sqrt{2} + \ln|\sqrt{2}+1|\right]$$

$$\pi\left[e\sqrt{1+e^2} - \ln|\sqrt{1+e}+e| - \sqrt{2} + \ln|\sqrt{2}+1|\right.$$

Pg 421

⑧

$y = \sin x$



$$2\pi \int_0^{\pi} y\sqrt{1+\cos^2 x}\,dx$$

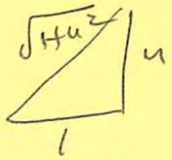
$$2\pi \int_0^{\pi} \sin x\sqrt{1+\cos^2 x}\,dx$$

Let $\cos x \overset{2}{=} u$ $\quad x = 0 \quad u = 1$

$-\sin x\,dx = du \quad x = \pi \quad u = -1$

$$-2\pi \int_1^{-1} \sqrt{1+u^2}\,du$$

$$2\pi \int_{-1}^{1} \sqrt{1+u^2}\,du$$



let $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$u = -1 \quad \theta = -\frac{\pi}{4}$

$u = 1 \quad \theta = \frac{\pi}{4}$

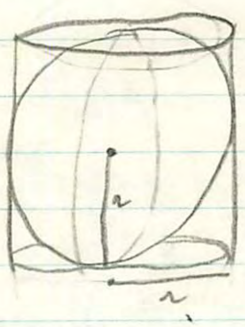
$$2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta = 14.42$$

$$2\pi \cdot \frac{1}{2}\left[\sec\theta\tan\theta - \ln(\sec\theta + \tan\theta)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\pi\left[\sqrt{2}\cdot 1 - \ln(\sqrt{2}+1) - \sqrt{2}(-1) + \ln(\sqrt{2}-1)\right]$$

$$\pi\left[2\sqrt{2} + \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

⑥



2π

Volume of Sphere $\frac{4}{3}\pi r^3$

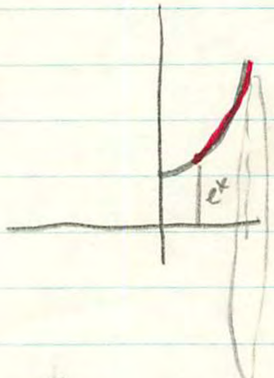
Volume of con $2\pi r^3$

a) $$\frac{\frac{4}{3}\pi r^3}{2\pi r^3} = \frac{2}{3}$$

b) Surface area of sphere is $4\pi r^2$
Surface area of con is $2\pi r \cdot 2R = 4\pi r^2$

Both are the same.

⑦



$y = e^x$
$y' = e^x$ $$2\pi\int_0^1 e^x\sqrt{1+e^{2x}}\,dx$$

Let $u = e^x$ $x=0$ $u=1$
$du = e^x\,dx$ $x=1$ $u=e$

$$2\pi\int_1^e \sqrt{1+u^2}\,du$$

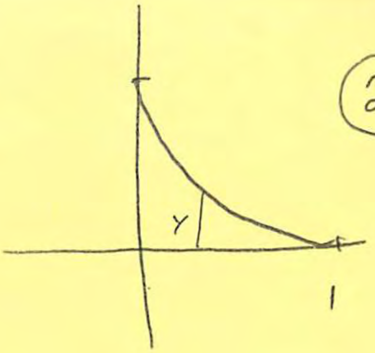
Let $u = \underline{\text{Tan}}\,\theta \Rightarrow \sec^3\theta$ Source tables to integrate

$$2\pi\cdot\frac{1}{2}\left[u\sqrt{1+u^2} + \ln\left|u+\sqrt{1+u^2}\right|\right]\Big|_1^e$$

$$\pi\left[e\sqrt{1+e^2} + \ln\left(e+\sqrt{1+e^2}\right) - \sqrt{2} - \ln(1+\sqrt{2})\right]$$

Pg 422

(22)



$$\underline{x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1}$$

$$\cancel{\tfrac{2}{3}} x^{-\frac{1}{3}} + \cancel{\tfrac{2}{3}} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

$$y^{\boxed{-\frac{1}{3}}} \frac{dy}{dx} = - x^{\boxed{-\frac{1}{3}}}$$

$$\frac{dy}{dx} = \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \therefore \left(\frac{dy}{dx}\right)^2 = \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}}$$

$$y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$2\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx$$

$$\sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}$$

$$2\pi \int \frac{\left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}}{x^{\frac{1}{3}}}\, dx$$

let $u = 1 - x^{\frac{2}{3}}$

$du = -\frac{2}{3} x^{-\frac{1}{3}}\, dx$

$-\frac{3}{2}\, du = \frac{dx}{x^{\frac{1}{3}}}$

Pg 421

$$y^{2/3} = 1 - x^{2/3}$$

22 $$x^{2/3} + y^{2/3} = 1$$

$$y = (1 - x^{2/3})^{3/2}$$

$$\cancel{\tfrac{2}{3}} x^{-\frac{1}{3}} + \cancel{\tfrac{2}{3}} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{1/3}}{x^{\frac{1}{3}}} \checkmark$$

$$\int 2\pi (1 - x^{2/3})^{3/2} \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}\, dx$$

$$\int 2\pi (1 - x^{2/3})^{3/2} \frac{\sqrt{x^{2/3} + y^{2/3}}}{x^{1/3}}\, dx = \int 2\pi (1 - x^{2/3})^{3/2} \frac{1}{x^{1/3}}\, dx$$

$$\int_0^1 2\pi \frac{(1 - x^{2/3})^{3/2}}{x^{\frac{1}{3}}}\, dx$$

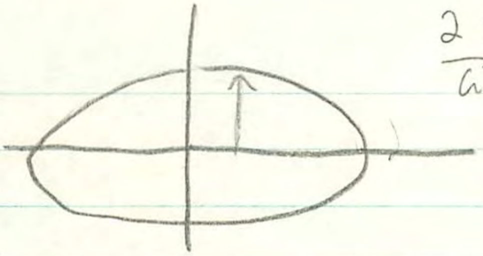
let $1 - x^{2/3} = u$ when $x = 0$ $u = 1$

$-\frac{2}{3} x^{-\frac{1}{3}} = du$ $x = 1$ $u = 0$

$$- -\frac{3}{2} \cdot 2\pi \int_0^1 u^{3/2}\, du = +3\pi \left(\frac{2}{5} u^{5/2}\right)\Big|_0^1 = \frac{+6\pi}{5}$$

Pg 421

㉔ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$\frac{2x}{a^2}\frac{dx}{dt}+\frac{2y}{b}=0$

$\frac{dx}{dt}=-\frac{ya^2}{xb^2}$

assume $a > b$

radius is $y=\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} = \frac{b}{a}\sqrt{a^2-x^2}$

$$y' = \frac{-2bx}{2a\sqrt{a^2-x^2}} = \frac{-bx}{a\sqrt{a^2-x^2}}$$

$$2\cdot 2\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}$$

$$\frac{4b\pi}{a}\int_0^a \cancel{\sqrt{a^2-x^2}}\,\frac{\sqrt{a^2(a^2-x^2)+b^2x^2}}{a\cancel{\sqrt{a^2-x^2}}}\,dx$$

$$\frac{4b\pi}{a^2}\int_0^a\sqrt{a^4-a^2x^2+b^2x^2}\,dx = \frac{4b\pi}{a^2}\int_0^a\sqrt{(a^2)^2-(a^2-b^2)x^2}\,dx$$

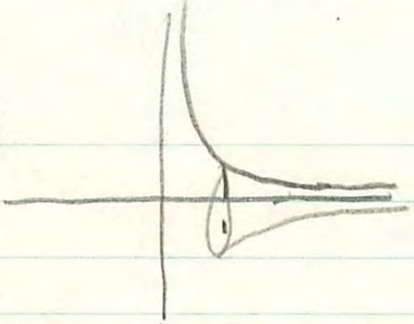
if $a=b$ ie a circle then we have

$$\frac{4\pi}{a}\int_0^a\sqrt{a^4}\,dx = \frac{4\pi}{a}(a^2x)\Big|_0^a = 4\pi a^2.$$

Pg 421

(25) $Y = \frac{1}{x}$

$Y' = -\frac{1}{x^2}$



1ST we consider the area under the curve.

$$A = \int_1^{\infty} y\,dx = \int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty} \Rightarrow \text{diverges ie infinite}$$

Next lets consider the volume of the rotation.
cross section is circle radius $\frac{1}{x}$

$$V = \pi\int_1^{\infty} \frac{dx}{x^2} = -\pi\left(\frac{1}{x}\right)\Big|_1^{\infty} = 0-\left(-\frac{\pi}{1}\right) = \pi \text{ finite}$$

finally we consider surface area

$$S = 2\pi\int_1^{\infty} y\sqrt{1+(y')^2}\,dx = 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx =$$

$$2\pi\int_1^{\infty} \frac{1}{x}\,\frac{\sqrt{x^4+1}}{x^2}\,dx = 2\pi\int_1^{\infty} \frac{\sqrt{x^4+1}}{x^3}\,dx$$

let $u = x^4+1$

$du = 4x^3\,dx$

$(u-1)^{3/4} = x^3$

$du = 4(u-1)^{3/4}\,dx$

$x=1 \quad u=2$

$x=\infty \quad u=\infty$

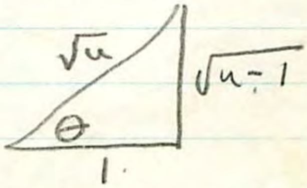
$$2\pi\int_2^{\infty} \frac{u^{1/2}\,du}{4(u-1)^{3/4}(u-1)^{3/4}}$$

$$\frac{\pi}{2}\int_2^{\infty} \frac{u^{1/2}\,du}{(u-1)^{3/2}}$$

let $u = \sec^2\theta \quad du = 2\sec^2\theta\tan\theta\,d\theta$

$u=2 \quad \theta=45 \quad u=\infty \quad \theta=\frac{\pi}{2} \quad \sqrt{u}=\sec\theta$

$$2\,\frac{\pi}{2}\int_{45}^{\frac{\pi}{2}} \frac{\sec^3\theta\tan\theta\,d\theta}{\tan^3} = \pi\int_{45}^{\frac{\pi}{2}} \frac{\sec^3\theta}{\tan^2\theta}\,d\theta =$$



$$\pi\int_{45}^{\frac{\pi}{2}} \frac{(\tan^2+1)\sec\theta\,d\theta}{\tan^2\theta} = \pi\int_{45}^{\frac{\pi}{2}} \sec\theta\,d\theta + \pi\int_{45}^{\frac{\pi}{2}} \frac{\sec\theta}{\tan^2}\,d\theta$$

prblm 25
cont

Pg 421

$$\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sec\theta\,d\theta + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sec\theta}{\tan^2\theta}\,d\theta \longrightarrow \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos\theta\,d\theta}{\sin^2\theta}$$

let $u = \sin\theta$
$du = \cos\theta\,d\theta$

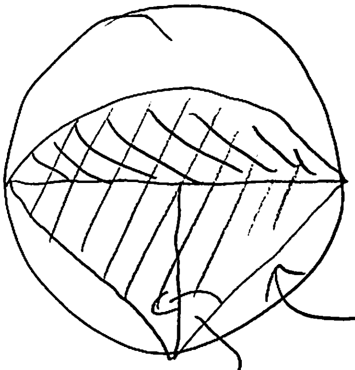
$$\lim_{b\to\frac{\pi}{2}} \pi\ln|\sec\theta+\tan\theta|\Big|_{\frac{\pi}{4}}^{b} + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos^2\theta}{\cos\theta\sin^2\theta}\,d\theta$$

$$\lim_{b\to\frac{\pi}{2}} \pi\ln|\sec\theta+\tan\theta|\Big|_{\frac{\pi}{4}}^{b} + \pi\int\frac{du}{u^2}$$

$$\lim_{b\to\frac{\pi}{2}} \pi\ln|\sec\theta+\tan\theta|\Big|_{\frac{\pi}{4}}^{b} - \frac{\pi}{\sin\theta}\Big|_{\frac{\pi}{4}}^{b}$$

$$\pi\lim_{b\to\frac{\pi}{2}}\Big(\underset{\Downarrow}{\ln}|\sec b+\underset{\Downarrow}{\tan b}| - \frac{1}{\sin b} - \ln|\sqrt{2}+1| + \sqrt{2}\Big)$$

Both diverge ie $\Rightarrow +\infty$ ∴ entire integral diverges ie surface area is infinite w/o calculus.



upper part = area of Sector − 2 triangles.

$$\frac{r^2\theta}{2} - 2\cdot\frac{1}{2}bh$$

radius = $(r\sqrt{2})^2\cdot\frac{\pi}{2}\cdot\frac{1}{2} - 2\cdot\frac{1}{2}r\cdot r$

angle $\frac{\pi}{2}$

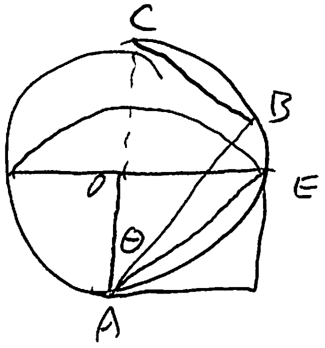
$$\frac{2r^2\pi}{4}$$

$\frac{r^2\pi}{2} - r^2 =$ upper part

Requested LUNE = Semi circle − upper part

$$\frac{\pi r^2}{2} - \left(\frac{\pi r^2}{2} - r^2\right)$$

$$r^2$$



AE = $r\sqrt{2}$

∠B is inscribed ∡ which is ½ central ∡.

∠B = 90° since ∡ AOC = 180°

we have the arc $\widehat{AC}$ common to both

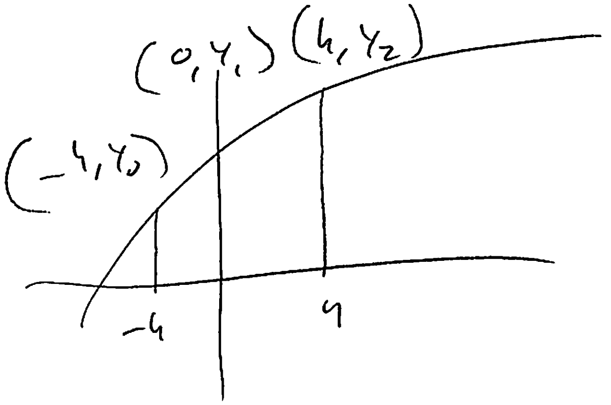
so $\cos\theta = \frac{AB}{AC} = \frac{AB}{2r} = \frac{Adj}{Hyp.}$

$$AB = 2r\cos\theta$$

$$2\int_0^{\frac{\pi}{4}} \frac{\overset{2}{\cancel{4}}r^2\cos^2\theta}{\cancel{2}} - \frac{\cancel{2}r^2}{\cancel{2}}\,d\theta$$

$$2r^2\int_0^{\frac{\pi}{4}} 2\cos^2\theta - 1\,d\theta.$$

$$\int_0^1 \sqrt{1-x^3}\,dx$$



$f(x) = ax^2+bx+c$

$$\int_{-h}^{h} ax^2+bx+c\,dx =$$

$$\left.\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right|_{-h}^{h} =$$

$$\frac{ah^3}{3}+\frac{bh^2}{2}+ch-\left(-\frac{ah^3}{3}+\frac{bh^2}{2}-ch\right)$$

$$\underbrace{\frac{ah^3}{3}+\cancel{\frac{bh^2}{2}}+ch+\frac{ah^3}{3}-\cancel{\frac{bh^2}{2}}+ch}$$

$$\frac{2a}{3}h^3+2ch$$

$$\underbrace{\frac{h}{3}\left(\underline{2ah^2}-6c\right)}$$

$$\frac{h}{3}\left(y_0+4ah^2+y_2\right)$$

$f(-h) = ah^2-bh+c = y_0$

$f(0) = c = y_1$

$f(h) = ah^2+bh+c = y_2$

~~$y_0+y_2 = 2ah^2+2c$~~

$y_0-y_1 = ah^2-bh$

$y_2-y_1 = ah^2+bh$

$y_0-2y_1+y_2 = \underline{2ah^2}$

← CORRECT. ~~good for CO! concave~~ down

pg 421-426

SIMPSON's Rule

1st read introduction of methods pg 421-423

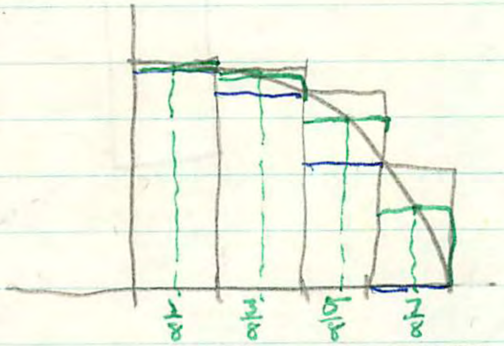
Calculator fnInt $(\sqrt{(1-x^3)}, x, 0, 1) =$ .841

eg1 $\int_0^1 \sqrt{1-x^3}\,dx$ approx using left endpt method

$N = 4$

$x_0 = 0 \quad x_1 = \frac{1}{4} \quad x_2 = \frac{1}{2} \quad x_3 = \frac{3}{4} \quad x_4 = 1 \quad h = \frac{1}{4}$

h is width of interval (rectangle)



$y = (1-x^3)^{\frac{1}{2}}$

$y' = \frac{1}{2}(1-x^3)^{-\frac{1}{2}}(-3x^2)$

$y'' = -\frac{6}{2}x^2(1-x^3)^{-\frac{1}{2}}$

$-3x^2\left(\frac{-1}{2}\right)(1-x^3)^{-3/2}(-3x^2)$

$y'' < 0$ for $x \in [0, 1)$

$\frac{1}{4}\left(\sqrt{1-0^3} + \sqrt{1-\left(\frac{1}{4}\right)^3} + \sqrt{1-\left(\frac{1}{2}\right)^3} + \sqrt{1-\left(\frac{3}{4}\right)^3}\right)$

$\frac{1}{4}(1 + .992 + .935 + .760) = \frac{1}{4}(3.688) = .922$

eg2 using Right end pts

$\frac{1}{4}\left(\sqrt{1-\left(\frac{1}{4}\right)^3} + \sqrt{1-\left(\frac{1}{2}\right)^2} + \sqrt{1-\left(\frac{3}{4}\right)^2} + 0\right)$ 1 less than so?

$\frac{1}{4}(.992 + .935 + .760) = \frac{1}{4}(2.688) = .672$

eg3 using mid pt

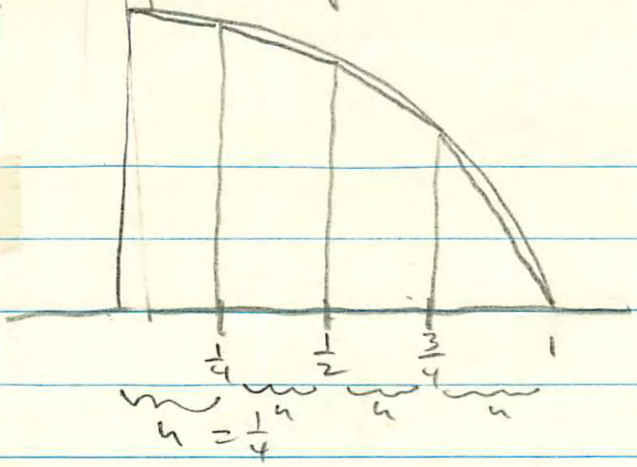
$\frac{1}{4}\left(\sqrt{1-\left(\frac{1}{8}\right)^3} + \sqrt{1-\left(\frac{3}{8}\right)^3} + \sqrt{1-\left(\frac{5}{8}\right)^3} + \sqrt{1-\left(\frac{7}{8}\right)^3}\right) =$

$\frac{1}{4}(.9990 + .9732 + .8694 + .5745) = \frac{1}{4}(3.416) =$ .8541

eg4 Trapezoidal method!

eg 4 using trapezoidal Rule

pg 421-426



$\frac{f(x_0)+f(x_1)}{2}h + h\frac{(f(x_1)+f(x_2))}{2}$ etc.

$\frac{h}{2}\left[f(x_0)+2f(x_1)+ \dots +2f(x_{n-1})+f(x_n)\right]$

$\frac{1}{4}\cdot\frac{1}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$

$\frac{1}{8}\left[\sqrt{1-0}+2\sqrt{1-(\frac{1}{4})^3}+2\sqrt{1-(\frac{1}{2})^3}+2\sqrt{1-(\frac{3}{4})^3}+0\right]$

$\frac{1}{8}\left[1+1.9843+1.8708+1.5207\right]=\frac{1}{8}(6.3758)=.7970$

★ Go To NextPg.

eg 5 using Simpsons method Divide [0,1] into 2n sections (n=2) of equal length $h=\frac{(1-0)}{4}=\frac{1}{4}$

$x_0=0 \quad x_1=0+\frac{1}{4} \quad x_2=\frac{1}{2} \quad x_3=\frac{3}{4} \quad x_4=1$

$h\cdot\frac{1}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right]$

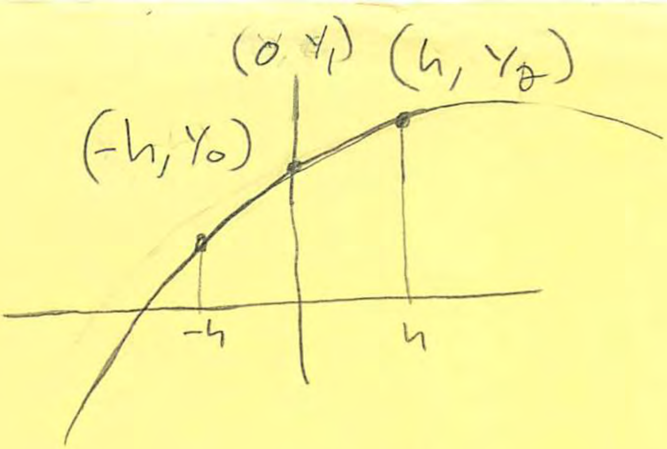
$\frac{1}{12}\left[\sqrt{1}+4\sqrt{1-(\frac{1}{4})^3}+2\sqrt{1-(\frac{1}{2})^3}+4\sqrt{1-(\frac{3}{4})^3}+\sqrt{0}\right]$

$\frac{1}{12}\left[1+3.9686+1.871+3.0414\right]=\frac{1}{12}(9.881)=.8234$

calculators answer $\text{fnInt}(\sqrt{1-x^3}, x, 0, 1) =$

.8413

$Y = ax^2 + bx + c$



Area under curve $= \int_{-h}^{h} ax^2 + bx + c \, dx$

$$= \frac{a}{3}x^3 + \frac{1}{2}bx^2 + cx \Big|_{-h}^{h}$$

$$= \frac{ah^3}{3} + \frac{b}{2}h^2 + ch - \left(-\frac{ah^3}{3} + \frac{bh}{2} - ch\right)$$

$$A = \frac{2ah^3}{3} + 2ch$$

$Y = ax^2 + bx + c$

$$= \frac{2ah^3 + 6ch}{3} = \frac{h}{3}(\underline{2ah^2} + 6c)$$

$x = -h \to \quad ah^2 - bh + c = Y_0$

$x = 0 \to \quad c = Y_1$ ✓

$x = h \to \quad ah^2 + bh + c = Y_2$

$Y_0 - Y_1 = ah^2 - bh$

$+ \quad Y_2 - Y_1 = ah^2 + bh$

$Y_0 - 2Y_1 + Y_2 = \underline{2ah^2}$

SAME

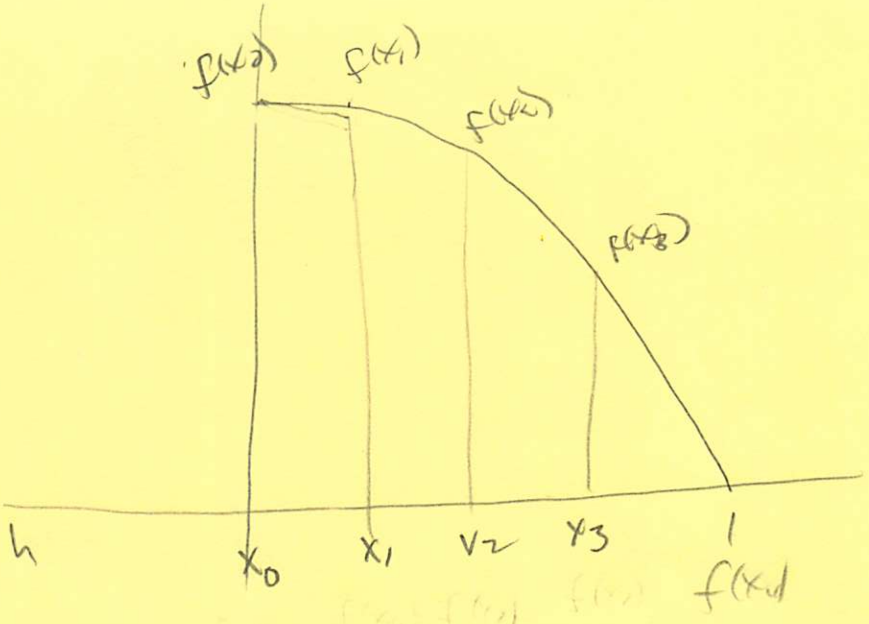
$c = Y_1$

So $A = \frac{h}{3}\left(Y_0 - 2Y_1 + Y_2 + 6Y_1\right)$

$\frac{h}{3}\left(Y_0 + 4Y_1 + Y_2\right)$

Simpson's method requires an even number of sections.

$$\int_0^1 \sqrt{1-x^3}\,dx$$



in all examples, $\frac{1}{4}$ is h
the width

Left $\frac{1}{4}f(x_0) + \frac{1}{4}f(x_1) + \frac{1}{4}f(x_2) + \frac{1}{4}f(x_3)$

$\frac{1}{4}\left(f(x_0) + f(x_1) + f(x_2) + f(x_3)\right))$

$+\frac{1}{4}f(x_1) + \frac{1}{4}f(x_2) + \frac{1}{4}f(x_3) + \frac{1}{4}(x_4)$

Ri t

$\frac{1}{4}\left(f(x_1) + f(x_2) + f(x_3) + f(x_4)\right))$

mid pt $\frac{1}{4}\left(f\left(\frac{x_1-x_0}{2}\right) + \frac{1}{4}f\left(\frac{x_2-x_1}{2}\right) + \frac{1}{4}f\left(\frac{x_3-x_2}{2}\right) + \frac{1}{4}f\left(\frac{x_4-x_3}{2}\right)\right)$

$\frac{1}{4}\left[f\left(\frac{x_1-x_0}{2}\right) + f\left(\frac{x_2-x_1}{2}\right) + f\left(\frac{x_3-x_2}{2}\right) + f\left(\frac{x_4-x_3}{2}\right)\right]$

Trapazoid $\frac{1}{4}\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \frac{f(x_2)+f(x_3)}{2} + \frac{f x_3 + f(x_4)}{2}\right)$

$\frac{1}{4}\cdot\frac{1}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$

Simpsons $\frac{1}{4}\cdot\frac{1}{3}\left(f(x_0) + 4f(x_1) + \underline{2f(x_2) + 4f(x_3)} + f(x_4)\right)$

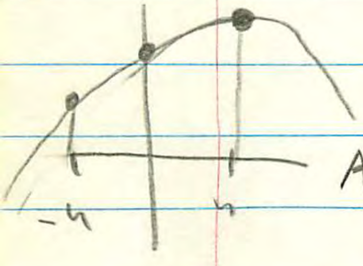
These become
the repeated
forms

before Demonstration of Simpson's method. Pg 421-424

Consider the Parabola $y = ax^2+bx+c$.

The area under the curve between $-h$ and h

is
$$A = \int_{-h}^{h} (ax^2+bx+c)\,dx = \frac{a}{3}x^3+\frac{1}{2}bx^2+cx \cancel{+C}\Big|_{-h}^{h}$$



$$\frac{a}{3}h^3 \cancel{+\frac{1}{2}bh^2} + ch + \frac{a}{3}x^3 \cancel{-\frac{1}{2}bh^2} + ch =$$

$$A = \boxed{\frac{2a}{3}h^3 + 2ch}$$

. Now this Parabola passes through $(-h, y_0)$, $(0, y_1)$ and (h, y_2)

So
$$y_0 = ah^2 - bh + c$$
$$y_1 = c$$
$$y_2 = ah^2 + bh + c$$

so
$$A = \frac{2ah^3 + 6ch}{3}$$

$$c = y_1$$

$$ah^2 - bh = y_0 - y_1$$
$$ah^2 + bh = y_2 - y_1$$
$$2ah^2 = y_0 + y_2 - 2y_1$$

$$A = \frac{h}{3}(2ah^2 + 6c)$$

$$A = \frac{h}{3}(y_0 + y_2 - 2y_1 + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Simpson's Rule is an application of the above. Simpson rule is to apply the formula to successive pieces of the curve between the interval a and b. each separate piece of the curve is approximated by an arc of a parabola through

Pg 421-
426

its ends and its midpt, The area under each parabolic arc is then given by an expression like ▲ and the results are summed to give:

$$A = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{N-2} + 4y_{N-1} + y_N\right]$$

Now go to Simpsons method at ★

After Demonstration of Simpson's method.:

Go back to Text pg 426 to discuss error of approximations.

Note $f^{(1)}(x)$ means 1st derivative of f
$f^{(4)}(x)$ means 4th derivative of f.

This appears logical but we are left without Justification. The references to the hw implies that we will establish some of this in our exercises, and the rest of it will come in chapter 15.

Consider: we want an approximation of $\ln(2) = .6931472$

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$ use 4 Subdivision

$$\frac{\text{length of interval}}{\text{\# of sections}} = \frac{2-1}{4} = \frac{1}{4} \text{ each.}$$

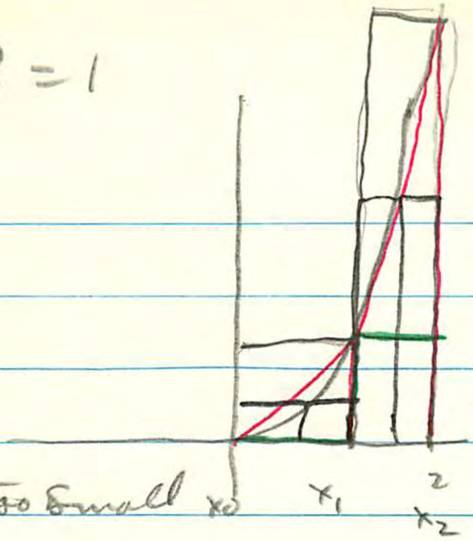
x	$\frac{1}{x}$	factor	Product
1	1	1	1
$\frac{5}{4}$	$\frac{4}{5} = .8$	4	3.2
$\frac{3}{2}$	$\frac{2}{3}$ $.66$	2	$1.3\overline{3}$
$\frac{7}{4}$	$\frac{4}{7} = .57$	4	2.28
2	$\frac{1}{2}$	1	$.5$
			8.31

$$\frac{h}{3} = \frac{\frac{1}{4}}{3} = \frac{1}{12}$$

$$\frac{1}{12} \cdot 8.31 \approx .69325$$

pg 427

① $\int_0^2 x^3\,dx$ $\quad h = \frac{2-0}{2} = 1$



Left PT method.

$1\left[0^3 + 1^3\right] = 1$ error 3 too small

actual area is $\frac{1}{4}x^4\Big|_0^2 = 4$

Rt pt method

$1\left[1^3 + 2^3\right] = 9$ error 5 too big

midpt method

$1\left[\left(\frac{1}{2}\right)^3 + \left(\frac{3}{2}\right)^3\right] = \frac{1}{8} + \frac{27}{8} = \frac{28}{8} = \frac{7}{2}$ error of $\frac{1}{2}$ too small

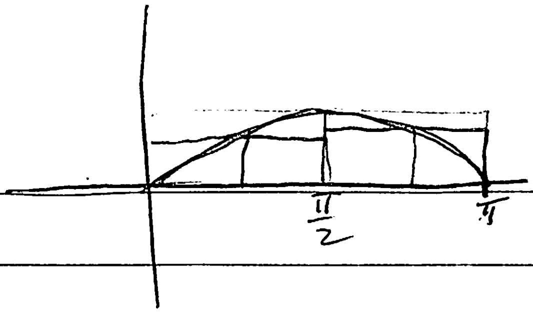
Trapezoidal method $\frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$

$\frac{1}{2}\left[0^3 + 2(1)^3 + 2^3\right] = \frac{1}{2}\left[10\right] = 5$ error of 1 too big.

Simpsons method.

$\frac{1}{3}\left[0^3 + 4(1)^3 + 2^3\right] = \frac{1}{3}(12) = 4$ this is exact since $f^{(4)}(x) = 0$.

② $\int_0^\pi \sin x\,dx$



Left End method

$\frac{\pi}{2}[0+1]=\frac{\pi}{2}$ Error $2-\frac{\pi}{2}$ Too small $-\cos x\Big|_0^\pi = 1+1=2$ Actual area.

Right end method.

$\frac{\pi}{2}[1+0]=\frac{\pi}{2}$ ERRor $2-\frac{\pi}{2}$ Too Small

Mid pt method

$\frac{\pi}{2}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right]=\frac{\sqrt{2}\,\pi}{2}$ ERRor $\frac{\pi\sqrt{2}}{2}-2$ Too big

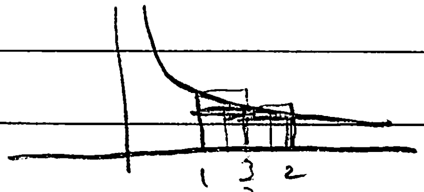
Trapezoid method,

$\frac{h}{2}=\frac{\pi}{4}$ $\frac{\pi}{4}[0+2(1)+0]=\frac{\pi}{2}$ Error $2-\frac{\pi}{2}$ Too Small

Simpson's method

$\frac{h}{3}$ $\frac{\pi}{6}[0+4(1)+0]=\frac{2\pi}{3}$ error $\frac{2\pi}{3}-2$ Too big.

⑬ $\int_1^2 \frac{dx}{x}$



Left end method

$\frac{1}{2}\left[1+\frac{2}{3}\right]=\frac{5}{6}$ error $\frac{5}{6}-\ln 2$ $\frac{5}{4}$ $\frac{7}{4}$ $\ln 2$ actual area.

RT end method

$\frac{1}{2}\left[\frac{2}{3}+\frac{1}{2}\right]=\frac{7}{12}$ error $\ln 2-\frac{7}{12}$

Mid pt method

$\frac{1}{2}\left[\frac{4}{5}+\frac{4}{7}\right]=\frac{1}{2}\left[\frac{28+20}{35}\right]\frac{1}{2}\cdot\frac{48}{35}=\frac{24}{35}$ $\ln 2-\frac{24}{35}$

TrapAzoid method

$\frac{1}{4}\left[1+2\left(\frac{2}{3}\right)+\frac{1}{2}\right]=\frac{1}{4}\left[\frac{6+8+3}{6}\right]=\frac{17}{24}$ $\frac{17}{24}-\ln 2$

Simpson's method

$\frac{1}{6}\left[1+4\left(\frac{2}{3}\right)+\frac{1}{2}\right]=\frac{1}{6}\left[\frac{6+16+3}{6}\right]=\frac{25}{36}$ $\frac{25}{36}-\ln 2$

⑲ $f(a) = f(b)$

Left Estimate is $h\left[f(x_0) + f(x_1) + \cdots + f(x_{N-1})\right]$

RT estimate is $h\left[f(x_1) + f(x_2) + \cdots + f(x_N)\right]$

but since $f(a) = f(b)$ $f(x_0) = f(x_1)$ ∴ Both the left and right estimates are equal.

Trapezoidal method is $\frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_N)\right]$

Since $f(x_0) = f(x_N)$ this becomes

$$\frac{h}{2}\left[2f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1})\right] =$$

$$h\left[f(x_0) + f(x_1) + \cdots + f(x_{N-1})\right.$$

which is the left estimate.

⑳ $\frac{\text{Left} + \text{RT}}{2} \overset{?}{=} \text{Trap.}$

$$\frac{1}{2}\left[h\left[f(x_0) + f(x_1) + \cdots + f(x_{N-1})\right] + h\left[f(x_1) + f(x_2) + \cdots + f(x_N)\right]\right]$$

$\frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_N)\right]$ = Trap method.

$f(x) = x^2$

(25) $N=1$ then $h = \frac{1-0}{1} = 1$ $x_0 = 0$ $x_N = 1$

Trap est. $\frac{1}{2}[f(0) + f(1)] = \frac{1}{2}[0^2+1^2] = \frac{1}{2}$

Actual $\frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$ so error is $\frac{1}{6}$

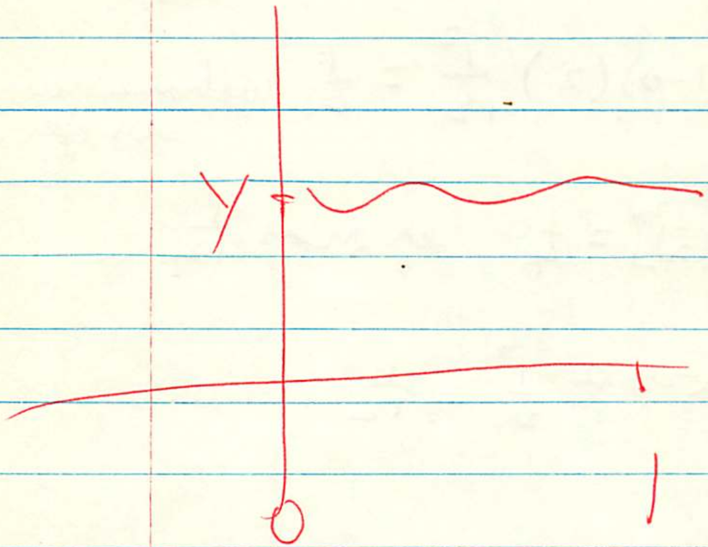
$f = x^2$ $f' = 2x$ $f''(x) = 2$ So 2 is max value of f''

$(b-a) M_2 \frac{h^2}{12} = (1-0)(2)\frac{1^2}{12} = \frac{1}{6}$ we have max error.

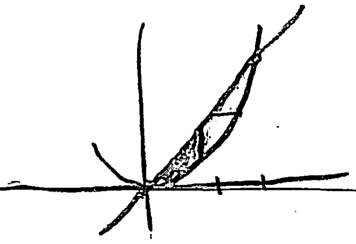
(26) midpt method $1\left[\left(\frac{1}{2}\right)^2\right] = \frac{1}{4}$ error is $\frac{1}{12}$.

$(b-a) M_2 \frac{h^2}{24} = (1-0)\, 2 \cdot \frac{1^2}{24} = \frac{1}{12}$

(28) Factors lecture about development of
(29) simpson's rule,


Y
0
1

Pg 432
G.Q.

① 

$y = x^2$ $\quad$ $y = 2x$

$\sqrt{y} = x$ $\quad$ $\frac{1}{2}y = x$

a) $\int_0^2 (2x - x^2)\,dx$ $\qquad$ b) $\int_0^4 (\sqrt{y} - \frac{1}{2}y)\,dy$

③ $y = \ln x$ $\quad$ $x \in [1, 2]$

$$\int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx = \int_1^2 \frac{\sqrt{x^2+1}}{x}\,dx$$

⑨ $x = \sin^2 t$ $\quad$ $0 \leq t \leq \frac{\pi}{2}$ $\quad$ $[0, 1]$ $\quad$ $x = 0 \quad t = 0$

$y = \cos^3 t$ $\quad$ $dx = 2\sin t \cos t\,dt$ $\quad$ $x = 1 \quad t = \frac{\pi}{2}$

$$\int_{x=a}^{x=b} y\,dx \Rightarrow 2\int_0^{\frac{\pi}{2}} \cos^4 t \sin t\,dt$$

let $u = \cos t$

$du = -\sin t\,dt$

$t = 0 \quad u = 1$

$t = \frac{\pi}{2} \quad u = 0$

$$-(-2)\int_0^1 u^4\,du = \frac{2}{5}u^5\Big|_0^1 = \frac{2}{5}$$

⑩ find $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}}$

$$\frac{\frac{dy}{dt} = -3\cos^{2} t \cancel{\sin t}}{\frac{dx}{dt} = 2\cancel{\sin t}\cos t} = -\frac{3}{2}\cos t\Big|_{t=\frac{\pi}{6}}$$

$$-\frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{4}$$

Pg 432
G.Q.

(11) $\int_1^{\infty} x^a\,dx$ convergent? for which a?

a) $\int_1^{\infty} x^a\,dx$ is convergent when $\lim_{b\to\infty}\int_c^b f(x)\,dx$ exist.

if $a>0$ $\lim_{b\to\infty} \left.\frac{x^{a+1}}{a+1}\right|_1^b$ diverges. (ie continues to grow larger)

if $a=0$ $\int_1^{\infty} dx = \lim_{b\to\infty} x^b$ agn diverges

if $-1<a<0$ $\int$ leads to x to a Positive power and agn diverges

if $a=-1$ $\int \Rightarrow \lim_{b\to\infty} \ln b$ agn diverges

if $\underline{\underline{a<-1}}$ $\int \Rightarrow \lim_{b\to\infty} x^{b+1}$ where $b+1<0$ → 0 converges.

b) $\int_0^{\infty} \frac{dx}{1+x^2+2x^4}$ $\lim_{b\to\infty}\int_1^b \frac{dx}{1+x^2+2x^4} < \lim_{b\to\infty}\int_1^b \frac{dx}{2x^4}$

$\lim_{b\to\infty}\frac{1}{2}\int_1^b x^{-4}\,dx = \lim_{b\to\infty} -\frac{1}{2}\frac{1}{3}x^{-3}\Big|_0^b = \lim_{b\to\infty} -\frac{1}{6b^3} \Rightarrow 0$

∴ since an integral larger than the one in question converges, the original converges

I think
This is OK.
we need to
(still) to show
that at x=0
we are OK.
we
with this?

$1+x^2+2x^4 > 1+x^2$ etc.

$\lim_{a\to 0}\int_a^1 \frac{dx}{1+x^2+2x^4} < \int_a^1 dx$

$\lim_{a\to 0} x\Big|_a^1 = \lim_{a\to 0} 1-a = 1$

Calculus Ch 10

"OBVIOUS", as applied to Calculus

Mr. Mumaugh

Pure Enjoyment # 2

Name ____________________________

This "honor quiz" is to be done by you ***totally* on your own**. You may use your text, notes and calculator. You are not allowed to discuss your solutions with anyone until the quiz has been turned in at the date and time shown on the homework schedule. Answer the questions on your own paper and staple your work to this sheet.

I

Given $x = t^2 + 5$, $y = t^3 - 9t$, for $-4 \le t \le 4$.

a) Sketch the curve with a window of $x \varepsilon [0,20]$ and $y \varepsilon [-20,20]$ T-Step = .2.

b) Determine the values of t and x when $y = 0$.

c) Find the slopes of the tangent lines at each of the **three** values of t found in part (b).

d) Using an integral in terms of t, find all points when the tangent line is horizontal.

e) Find the area enclosed by the loop. (Hint: explore the curve using the TRACE command before attempting this problem.

f) Sames as part "e" above except ... "in terms of x" etc.

II

We use $x = r\cos\theta$ and $y = r\sin\theta$ as a means to convert back and forth between cartesian and polar coordianates. Given: $r = a\sin\theta + b\cos\theta$.

a) Use the given and rewrite x and y so the variable r is eliminated.

b) Using the results of part a, find $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}}$.

Do not use a calculator to determine this.

c) Convert r to cartesian form.

d) Determine the cartesian coordinates of the center of the circle and specifically state the radius.

e) Using the results of part c, find $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}}$.

Do not use a calculator to determine this.

	θ	r
1	0	b
2	$\frac{\pi}{6}$	$\frac{a}{2} + b\frac{\sqrt{3}}{2}$
3	$\frac{\pi}{4}$	$a\frac{\sqrt{2}}{2} + b\frac{\sqrt{2}}{2}$
4	$\frac{\pi}{3}$	
5	$\frac{\pi}{2}$	
6	$\frac{2\pi}{3}$	
7	$\frac{3\pi}{4}$	
8	$\frac{5\pi}{6}$	
9	π	$-b$
10	$\frac{7\pi}{6}$	$-\frac{a}{2} - b\frac{\sqrt{3}}{2}$

f) Copy and fill in the table on the right:

g) What is the significance of line 10 (as compared to line 2), for example?

h) Find the area bounded by the graph of r. Use the formula for the area bounded by the graph of a polar function. (Hint: The information gained in part g is vital to this problem. Use it wisely.)

i) Using the formula for the length of a piece of the graph of a polar function, find the circumference of the figure.

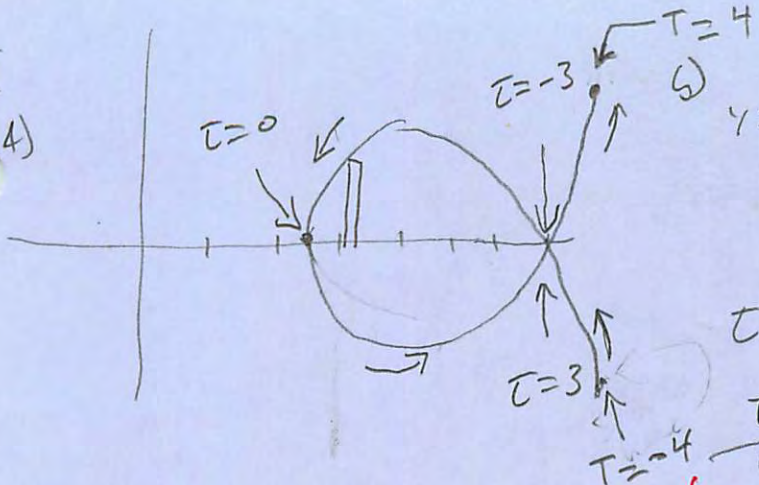
j) Using IIc and appropriate cartesian formulas, verify your answers to IIh and IIi.

The work on this problem was done by me with no assistance from another person ________________________

your signature

I

4)



b) $y = \tau^3 - 9\tau \overset{\text{set}}{=} 0 \qquad x = \tau^2 + 5$

$\tau(\tau^2 - 9) = 0$

$\tau = 0 \qquad \tau = \pm 3$

τ	x	y
0	5	0
3	14	0
-3	14	0

c) $\left.\frac{dy}{dx}\right|_{\tau \in \{0, \pm 3\}}$

$y = \tau^3 - 9\tau \qquad x = \tau^2 + 5$

$\frac{dy}{d\tau} = (3\tau^2 - 9) \qquad \frac{dy}{dt} = 2\tau$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \left.\frac{3\tau^2 - 9}{2\tau}\right|_{\tau = 0 \pm 3}$

$\frac{18}{6} \quad \frac{dy}{dt} = 3$ for $\tau = 3$ $\qquad \frac{dy}{dt}$ DNE for $\tau = 0$.

$\frac{18}{-6} \quad \frac{dy}{dt} = -3$ for $T = -3$

$x = \tau^2 + 5 \qquad y = \tau^3 - 9\tau$

$\tau(\tau^2 - 9)$

$\pm\sqrt{3}\ (-6)$

d) $\frac{dy}{dt} \overset{\text{set}}{=} 0 \qquad \frac{3\tau^2 - 9}{2\tau} \overset{\text{set}}{=} 0$

$3T^2 - 9 = 0 \qquad \tau = \sqrt{3} \quad (8, -6\sqrt{3})$

$\tau^2 = 3 \qquad \tau = -\sqrt{3} \quad (8, 6\sqrt{3})$

$\tau = \pm\sqrt{3}$

e) $2\int_{\tau=0}^{T=3} Y\,dx = 2\int_{\tau=0}^{\tau=-3} Y \cdot \frac{dx}{dt} \cdot dt = 2\int_{\tau=0}^{\tau=-3} (\tau^3 - 9\tau)\, 2\tau\, dt =$

$4\int_{\tau=0}^{\tau=-3} \tau^4 - 9\tau^2\, d\tau = 4\left(\frac{\tau^5}{5} - 3\tau^3\right)\Big|_0^{-3} = 4\left(\frac{-243}{5} + 81\right) = \frac{648}{5} = 129.6$

$$\begin{array}{r} 405 \\ 243 \\ \hline 162 \\ 4 \\ \hline 648 \end{array}$$

II $x = r\cos\theta \quad y = r\sin\theta \quad r = a\sin\theta + b\cos\theta$.

a) $x = a\overset{u}{\sin\theta}\,\overset{v}{\cos\theta} + b\cos^2\theta \qquad y = a\sin^2\theta + b\,\overset{u}{\sin\theta}\,\overset{v}{\cos\theta}$

b) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad \dfrac{dy}{d\theta} = 2a\sin\theta\cos\theta + b\cos^2\theta - b\sin^2\theta$

$$\frac{dx}{d\theta} = a\cos^2\theta - a\sin^2\theta - 2b\sin\theta\cos\theta$$

$$\frac{dy}{dx} = \left.\frac{2a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta)}{a(\cos^2\theta - \sin^2\theta) - 2b\sin\theta\cos\theta}\right|_{\theta=\frac{\pi}{2}}$$

$$= \frac{2a(1\cdot 0) + b(0-1)}{a(0-1) - 2b\cdot(1)(0)} = \frac{-b}{-a} = \frac{b}{a}$$ II B

c) $r^2 = ar\sin\theta + br\cos\theta$

$x^2 + y^2 = ay + bx$

d) $x^2 - bx + \frac{b^2}{4} + y^2 - ay - \frac{a^2}{4} = \frac{a^2+b^2}{4}$

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{\sqrt{a^2+b}}{2}\right)^2$$

center $\left(\frac{b}{2}, \frac{a}{2}\right)$ radius $\frac{\sqrt{a^2+b^2}}{2}$

e) $2x + 2y\frac{dy}{dx} = a\frac{dy}{dx} + b$

when $\theta = \frac{\pi}{2}$ $x = 0$ $y = r$ and $r\left(\frac{\pi}{2}\right) = a$

$(2y - a)\frac{dy}{dx} = b - 2x$

$\frac{dy}{dx} = \frac{b - 2x}{2y - a} \rightarrow \frac{b - 0}{2a - a} = \frac{b}{a}$ same as IIb, as you would expect.

II f

	θ	r	
1	0	b	
2	$\frac{\pi}{6}$	$\frac{a\sqrt{2}}{2}+\frac{b\sqrt{2}}{2}$	
3	$\frac{\pi}{4}$	$\frac{a\sqrt{2}}{2}+\frac{b\sqrt{2}}{2}$	
4	$\frac{\pi}{3}$	$\frac{a\sqrt{3}}{2}+b$	*
5	$\frac{\pi}{2}$	a	*
6	$\frac{2\pi}{3}$	$\frac{a\sqrt{3}}{2}-\frac{b}{2}$	*
7	$\frac{3\pi}{4}$	$\frac{a\sqrt{2}}{2}-\frac{b\sqrt{2}}{2}$	*
8	$\frac{5\pi}{6}$	$\frac{a}{2}-\frac{b\sqrt{3}}{2}$	*
9	π	$-b$	$= r(0)$
10	$\frac{7\pi}{6}$	$-\frac{a}{2}-\frac{b\sqrt{3}}{2}$	$= -\left(\frac{a}{2}+\frac{b\sqrt{2}}{2}\right) = r\left(\frac{\pi}{6}\right)$

II g) Starting with line 9, we are repeating the same points. The significance of this is that the entire figure is drawn when $0 \le \theta \le \pi$

II h) area $\int_{\alpha}^{\beta} \frac{r^2}{2}\,d\theta \rightarrow \frac{1}{2}\int_0^{\pi} (a\sin\theta + b\cos\theta)^2\,d\theta$

$$= \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \frac{a^2}{2}(1-\cos 2\theta) + \frac{b^2}{2}(1+\cos 2\theta) + 2ab\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\cdot\theta - \frac{a^2}{4}\sin 2\theta + \frac{b^2}{2}\theta + \frac{b^2}{4}\sin 2\theta + \frac{2ab\sin^2\theta}{2}\right]\Big|_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\pi - 0 + \frac{b^2\pi}{2} + 0 + 0 - 0\right] = \frac{\pi}{4}(a^2+b^2)$$

II i) $s(\theta) = \int_{\alpha}^{\theta} \sqrt{r^2+\dot{r}^2}\,d\theta$

$r = a\sin\theta + b\cos\theta$

$r^2 = a^2\sin^2\theta + ab\sin\theta\cos\theta + b^2\cos^2\theta$

$\dot{r} = a\cos\theta - b\sin\theta$

$\dot{r}^2 = a^2\cos^2\theta - 2ab\sin\theta\cos\theta + b^2\sin^2\theta$

$$= \int_{\alpha}^{\theta} \sqrt{a^2+b^2}\,d\theta = \sqrt{a^2+b^2}\,\theta\Big|_0^{\pi} = \pi\sqrt{a^2+b^2}$$

II_J area of circle πr^2

$$\left(x-\frac{b}{2}\right)^2+\left(y-\frac{a}{2}\right)^2=\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

$\therefore\ \pi\left(\frac{a^2+b^2}{4}\right)$ same as II_h

circumference of circle $2\pi r$

$$\frac{\cancel{2}\pi\sqrt{a^2+b^2}}{\cancel{2}}=\pi\sqrt{a^2+b^2}$$

same as $\text{II}\,i$

DOWNLOAD

Jim Mumaugh - Vaca High
From: Jim [teeitup98@telocity.com]
Sent: Wednesday, November 27, 2002 7:51 AM
To: Jim Mumaugh - Vaca High
Subject: FW: [ap-calculus] Re: try this application problem with your students

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From: **"J" <jconlee@cgs.k12.va.us>**
Reply-To: "AP-Calculus" <ap-calculus@lyris.collegeboard.com>
Date: Tue, 26 Nov 2002 10:20:29 -0800
To: "AP-Calculus" <ap-calculus@lyris.collegeboard.com>
Subject: [ap-calculus] Re: try this application problem with your students

Here are "my" answers, graded on a 9 point rubric. Enjoy

1. James recently downloaded a new song using his cable modem and the Internet. The table below shows the download speeds (kilobytes/sec) on 5-second intervals.

Time, t, seconds	Kilobytes, kb
0	0
5	7.03
10	18.25
15	47.82
20	61.18
25	34.82
30	15.75
35	16.28
40	45.15
45	123.95
50	115.09
55	44.64
60	24.05

a) Using 6 evenly partitioned rectangles, compute the midpoint Riemann sum that approximates the total size of the downloaded song.

Using the midpoint y-values of the 6 intervals (which occur at t=5,15,25,35,45, and 55), I summed them up and multiplied by 5, the length (or width of each interval).
5*(7.03+47.82+34.82+16.28+123.95+44.64)□72.7 kb or the approximate size of this file is 1372 kb using MRAM with 6 equal intervals

2 points-1 for setup and 1 for answer (with units)

T=40

b) Estimate how fast the download is accelerating at time t@ seconds. Show setup and use appropriate units.

My preferred method is to take the averate rate of change using 35 and 45 seconds. The change in my numerator is 123.95-16.28□7.67 and then divide by (45-35) yields an approximation that tells me that at the time t@ s, the download speed is accelerating at 10.767 kb/sec per sec. There are other ways to get an answer for this section but

Jim Mumaugh - Vaca High

From: Jim [teeitup98@telocity.com]
Sent: Monday, November 25, 2002 2:47 PM
To: Jim Mumaugh - Vaca High
Subject: FW: [ap-calculus] try this application problem with your students

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From: **"J" <jconlee@cgs.k12.va.us>**
Reply-To: "AP-Calculus" <ap-calculus@lyris.collegeboard.com>
Date: Sun, 24 Nov 2002 07:02:37 -0800
To: "AP-Calculus" <ap-calculus@lyris.collegeboard.com>
Subject: [ap-calculus] try this application problem with your students

Dear all, I recently had fun developing this question, which should provide most students with a relevant application to integrals and calculus.

1. James recently downloaded a new song using his cable modem and the Internet. The table below shows the download speeds (kilobytes/sec) on 5-second intervals.

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45	123.95
50	115.09
55	44.64
60	24.05

5 · 7.03 +

10225.7

a) Using 6 evenly partitioned rectangles, compute the midpoint Riemann sum that approximates the total size of the downloaded song.

b) Estimate how fast the download is accelerating at time t@ seconds. Show setup and use appropriate units.

c) James later found a function y=(x+5)*e^sin(x/4-lnx) that models the download speeds for his computer. If 85 kilobytes were downloaded during the first ten seconds, use the function to approximate the total size of the file if it took ~~70~~ seconds to complete the download.
60

James recently downloaded a new song using his cable modem and the Internet. The table below shows the download speeds (kilobytes/sec) on 5-second intervals.

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a) Using 6 evenly partitioned rectangles, compute the midpoint Riemann sum that approximates the total size of the downloaded song.

b) Estimate how fast the download is accelerating at time t@ seconds. Show setup and use appropriate units.

c) James later found a function y=(x+5)*e^sin(x/4-lnx) that models the download speeds for his computer. If 85 kilobytes were downloaded during the first ten seconds, use the function to approximate the total size of the file if it took ~~70~~ seconds to complete the download.

60

d) Using your answer found in part c), find the average download speed for James' computer on the interval [0,60]. Show setup and use calculator to evaluate. Explain the meaning of your answer.

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7.03
47.82
34.82
16.28
123.95
64.64
+

274.54 *10 = ~~1372.7~~ 2745.4

so The Approximate total download is about ~~1373~~ Kb.
2745.4 Kb

b) at t = 40 seconds.

$$\frac{123.95 - 16.28}{10} = 10.767.$$

accelerating at about 11 mb/sec.

c)

$$y = (x+5)e^{\sin\left(\frac{x}{4} - \ln x\right)}$$

$$\int_0^{10} y\,dx + \int_{10}^{60} (x+5)\,e^{\sin\left(\frac{x}{4} - \ln x\right)}\,dx$$

85 + 2642.281

2727.281

d)

$$\frac{2727.281}{60} \approx 45.\ \text{Kb/sec.}$$

Summary
Pg 432

① Average value of a fct'n over an interval [a,b]

a) is defined $\frac{1}{b-a}\int_a^b f(x)\,dx$.

b) See inset left column pg 381. The height of the rectangle is the Average of f(x). The rectangle has the same area as that under the curve,

② There are 3 formulas for arc length

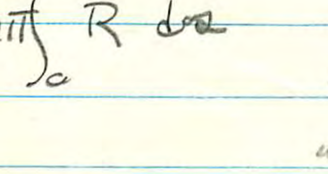
a) $\int_a^b \sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$ when given in a parameter

b) $\int_a^b \sqrt{1+(y')^2}\,dx$ when given f(x) = or y = .

c) $\int_a^b \sqrt{r^2+(r')^2}\,d\theta$ when given polar coord.

③ area in Polar coord $\int_\alpha^\beta \frac{r^2}{2}\,d\theta$

Think of the area as that of a triangle where r dθ is the altitude.



$\frac{1}{2}\, r \cdot r\, d\theta$

④ Surface area of revolution,

$2\pi\int_a^b R\,ds$ where 2π revolves. R is radius of revolution and $\int ds$ is length of the line being rotated.

Pg 432-433

⑥ $\int_a^b f(x)\,dx$ can be ~~estimated~~ w/out error

a) by midpt method if $f''=0$ linear

b) by Trap method. if $f''=0$ linear

c) by Simpsons method if $f'''=0$ cubic.

⑦

x	1	1.5	2	2.5	3	3.5	4
f(x)	4	5	6	8	9	11	14
	①	②	③	④	⑤	⑥	

$\int_1^4 f(x)\,dx$ estimate using Simpson's method

$h = \frac{1}{2}$ (width of each interval)

$x_0 = 1$ $x_1 = 1.5$ etc There are 6 interval. ∴ $n=6$

$$\frac{1}{2}\cdot\frac{1}{3}\left[4 + 4(5) + 2(6) + 4(8) + 2(9) + 4(11) + 14\right]$$

$$\frac{1}{6}\left[144\right] = 24.$$

⑧

Time	1	2	3	4	5	6	7
Temp	81	75	80	83	78	70	59
	①	②	③	④	⑤	⑥	

$N=6$ $h=1$

$x_0 = 81$ etc.

$$\frac{1}{3}\left[81 + 4\cdot 75 + 2\cdot 80 + 4\cdot 83 + 2\cdot 78 + 4\cdot 70 + 59\right] = \frac{1}{3}(1368)$$

456

Average Temp then is $\frac{1}{7-1}\int_1^7 f(x)\,dx = \frac{1}{6}\cdot 456 = 76^0$

433
Summary

left or right ⑨ a) $\frac{(b-a)M_1 h}{2}$ if you double the # of sections you cut h in half this would cut the error by $\frac{1}{2}$

midpt b) $\frac{(b-a)M_2 h^2}{24}$ if you double the # of sections you cut h in half but it is squared so error will be $\frac{1}{4}$ as much.

trap c) $\frac{(b-a)M_2 h^2}{12}$ agn error is cut $\frac{1}{4}$ of original

Simpsons d) $\frac{(b-a)M_4 h^4}{180}$ here $\left(\frac{h}{2}\right)^4 = \frac{h}{16}$ so error is $\frac{1}{16}$ of original

433
Summary

⑬ $\int_0^{\infty} \frac{dx}{(x-1)^2}$ Since $x=1$ is Not allowed;

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^2} + \lim_{a\to 1^+}\int_a^2 \frac{dx}{(x-1)^2} + \lim_{b\to\infty}\int_2^b \frac{dx}{(x-1)^2}$$

$$\Rightarrow = \lim_{b\to 1^-}\left(\frac{-1}{x-1}\right)\Big|_0^b = \lim_{b\to 1^-}\left(\frac{-1}{b-1} - \frac{-1}{-1}\right) =$$

$\lim_{b\to 1^-}\left(\frac{1}{1-b} - 1\right)$ This diverges since $b \to 1^-$ and $1-b \to 0^+$ so $\frac{1}{1-b} \Rightarrow +\infty$.

Since atleast one portion diverges the entire integral diverges.

? ⑭ $\int_0^{\infty} e^{-x}\sin(2x+3)\,dx$ $\lim_{b\to\infty}\int_0^b e^{-x}\sin(2x+3)\,dx$

$u = e^{-x}$ $\sin(2x+3)$
$+$
$-e^{-x}$ $2\cos(2x+3)$
$-$
e^{-x} $+$ $-4\sin(2x+3)$

$$= 2e^{-x}\cos(2x+3) - 4e^{-x}\sin(2x+3) - 4$$

$$= \lim_{b\to\infty} \frac{1}{5}\left[2e^{-x}\cos(2x+3) - 4e^{-x}\sin(2x+3)\right]$$

$$\lim_{b\to\infty} \frac{2}{5}\left[\frac{\cos(2x+3) - 2\sin(2x+3)}{e^x}\right]$$

$$\lim_{b\to\infty} \frac{2}{5}\left[\frac{\cos 2x\cos 3 - \sin 2x\sin 3 - 2\sin 2x\cos 3 + 2\sin 3\sin 2x}{e^x}\right]$$

$$\lim_{b\to\infty} \frac{2}{5}\left[-\sin 2x(\sin 3 + 2\cos 3) + \cos 2x(\cos 3 - 2\sin 3)\right]$$

p, 433
Summary

? ⑮ $\Gamma(x) = \int_0^\infty e^{-\tau} \tau^{x-1}\, d\tau \quad x > 0$

a) $\int_0^b e^{-\tau} \tau^{x-1}\, d\tau \quad e^{-\tau} < 1 \therefore \quad e^{-\tau}\tau^{x-1} < \tau^{x-1}$

$\int_0^b e^{-\tau}\tau^{x-1}\, d\tau < \int_0^b \tau^{x-1}\, d\tau$

$\lim_{b \to \infty} \int_0^b \tau^{x-1}\, d\tau = \lim_{b \to \infty} \left. \frac{\tau^x}{x} \right|_0^b = \lim_{b \to \infty} \frac{b^x}{x}$

too hard doesn't make sense.

433
Summary

⑯ $\int_0^1 \frac{\ln x}{1-x^2}\,dx$ ftn not defined at 0 or 1.

$$\lim_{x\to 1^-} \frac{\ln x}{1-x^2} = \lim_{x\to 1^-} \frac{\frac{1}{x}}{-2x} = \lim_{x\to 1^-} \frac{-1}{2x^2} = \frac{-1}{2}$$

So define the integrand to be $-\frac{1}{2}$ for $x=1$ and we have a continuous fctn. So only pblm will be at $x=0$. Pick a constant, $\frac{1}{2}$, for example $1-x^2 > \frac{1}{2}$, $\frac{1}{2} > x^2$

$-x < \frac{1}{\sqrt{2}}$

so $0 < \frac{1}{1-x^2} \leq 2$ for $0 \leq x \leq \frac{1}{\sqrt{2}}$

$0 \leq \left|\frac{\ln x}{1-x^2}\right| \leq 2|\ln x|$ for the same interval Notice that $\ln x$ will be <0 so

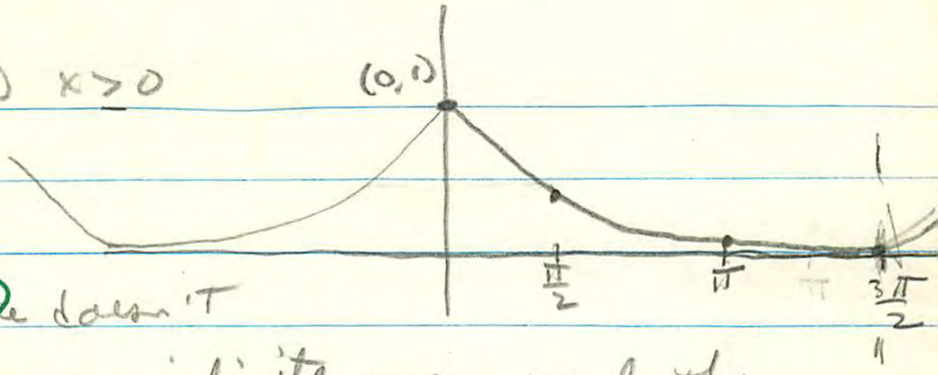
$$-\frac{\ln x}{1-x^2} \leq -2\ln x$$

So what remains is to show that we can integrate $-2\ln x$ then we can integrate $\frac{-\ln x}{1-x^2}$ then we could also integrate $\frac{+\ln x}{1-x^2}$.

β 433

⑰

$y = e^{-x}(1+\sin x) \quad x \geq 0$



Curve doesn't cross

x axis so our line doesn't ?

end. we will have infinite area under the

curve ∴ infinite for revolution also.

See below, $y=0$ when $x = \frac{3\pi}{2}$

~~$2\pi \int x \sqrt{1+x'} \, dx$~~

~~$2\pi \int x$~~

$y = \overset{u}{e^{-x}}\overset{v}{(1+\sin x)}$

~~$\frac{dy}{dx} = -(1+\sin x)e^{-x} + e^{-x}\cos x$~~

$= -e^{-x}(1+\sin x - \cos x)$

START again!

WTS

$$\int_0^{\infty} e^{-x}(1+\sin x)\,dx$$

divrgs. ⇒ ∞ ∴ infinite area below so we will have infinite volume upon rotation.

$$\int_0^{\infty} e^{-x}dx + \int_0^{\infty} e^{-x}\sin x \, dx$$

$$\int_a^b e^{-x}dx = -e^{-x}\Big|_a^b = -e^{-b}+e^{-a}$$

$$\lim_{\substack{a\to 0\\ b\to\infty}}(-e^{-b}+e^{-a}) = \lim_{\substack{a\to 0\\ b\to\infty}} \frac{1}{e^a} - \frac{1}{e^b} = 1-0=1$$

$$\int_a^b \overset{u}{e^{-x}}\,\overset{dv}{\sin x\,dx} = \boxed{-\cos x\, e^{-x}} - \int e^{-x}\cos x\,dx$$

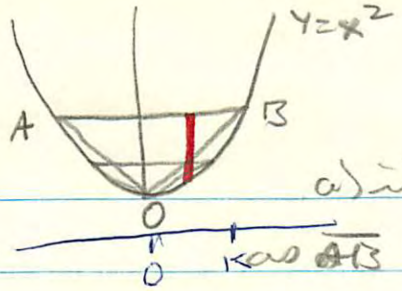
$u = e^{-x}$ $\quad v = -\cos x$

$du = -e^{-x}$ $\quad dv = \sin x\,dx$

$\lim_{x\to\infty}$ of ↑ is not defined ∴ original does not converge ∴ infinite area.

433-434
Summary

(25)



a) it appears the Δ has less area but as $\overline{AB}$ lowers to zero the ratio approach

B) let $\overline{AB} = Y = K^2$ So coordinates of $A = (-K, K^2)$, $B = (K, K^2)$ Δ AOB has area $\frac{1}{2}(2K)(K^2) = K^3$.

area inside Parabola $= 2\int_0^K (K^2 - x^2)\,dx = 2K^2x - \frac{2}{3}x^3\Big|_0^K =$

$2K^3 - \frac{2}{3}K^3 = \frac{4K^3}{3}$ $\qquad \frac{\text{Area } \Delta}{\text{Area Sector}} = \frac{K^3}{\frac{4K^3}{3}} = \frac{3}{4}$

This means that where ever the line AB is the ratio is a constant 3/4 !!

P 434
Summary

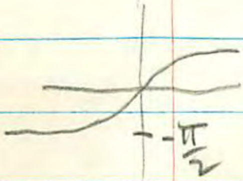
㉘ $G(a) = \int_0^{\infty} \frac{a}{1+a^2x^2}\,dx$

a) $G(0) = \int_0^{\infty} \frac{0}{1}\,dx = 0$

b) $a<0$ $\quad \lim_{b\to\infty} \int_0^b \frac{a}{1+a^2x^2}\,dx$

let $u = ax$
$du = a\,dx$
$x=0 \quad u=0$
$x=\infty \quad u=-\infty$

$$G(a) = \int_0^{-\infty} \frac{du}{1+u^2} = -\int_{-\infty}^{0} \frac{du}{1+u^2} = \lim_{b\to-\infty} \int_b^0 \frac{du}{1+u^2} =$$



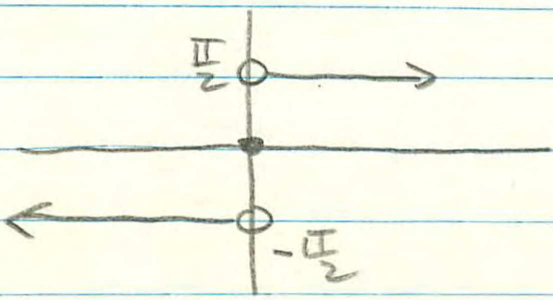
$$\lim_{b\to\infty} -\text{Tan}^{-1} u \Big|_b^0 = -\lim_{b\to-\infty} \left(-\text{Tan}^{-1} 0 - \text{Tan}\, b\right) =$$

$$-0 + \text{Tan}^{-1}(-\infty) = -\frac{\pi}{2}$$

c) $a>0$ Same except positive $\lim_{b\to\infty} -\text{Tan}^{-1} u \Big|_b^0 =$
The neg in front of Tan^{-1} is a
result of switching limits of integrations

$$\lim_{b\to\infty} \left(-\text{Tan}\, 0 + \text{Tan}\, b\right) = \frac{\pi}{2}$$

So graph of $G(a)$ $\begin{cases} \frac{\pi}{2} & \text{if } a>0 \\ 0 & \text{if } a=0 \\ -\frac{\pi}{2} & \text{if } a<0 \end{cases}$



Pg 434

(30) a) $\int \frac{\ln x}{\sqrt{x}}\,dx$ let $u^2 = x$

$2u\,du = dx$

$$\int \frac{2\ln u\,(2\cancel{u})\,du}{\cancel{u}} = 4\int \ln u\,du$$

$a = \ln u$ $b = u$

$da = \frac{1}{u}\,du$ $db = du$

$$4\left[u\ln u - \int du\right] = 4u\ln u - 4u = 4u(\ln u - 1)$$

$$4\sqrt{x}\,(\ln\sqrt{x} - 1) = 4\sqrt{x}\left(\tfrac{1}{2}\ln x - 1\right) = 2\sqrt{x}(\ln x - 2)$$

b) $\int \sqrt{\ln x}\,dx$ I can't see any reasonable substitution. We hope it is the one which is not elementary.

c) $\int \frac{\sqrt{\ln x}}{x}\,dx$ let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{3/2} = \frac{2}{3}(\ln x)^{3/2}$$

Pg 434
Summary

(32) No fctn not defined for $x = -\frac{1}{2}$

$$\int_{-2}^{1} \frac{dx}{2x+1} = \lim_{b \to -\frac{1}{2}^-} \int_{-2}^{b} \frac{dx}{2x+1} + \lim_{a \to -\frac{1}{2}^+} \int_{a}^{1} \frac{dx}{2x+1}$$

$$\lim_{b \to -\frac{1}{2}^-} \left(\frac{1}{2} \ln|2x+1|\right)\Big|_{-2}^{b} \Rightarrow -\infty$$ ∴ fctn diverges.

? (33)

$$\int \sqrt{\frac{1}{(1+\theta)^2} + \frac{1}{(1+\theta)^4}}\, d\theta \qquad r' = \frac{-1}{(1+\theta)^2} \qquad r = \frac{1}{1+\theta}$$

$$\int \sqrt{\frac{(1+\theta)^2+1}{(1+\theta)^4}}\, d\theta = \int_0^\infty \frac{\sqrt{2+2\theta+\theta^2}}{(1+\theta)^2}\, d\theta$$

$$\lim_{b \to \infty} \int_0^\infty$$

let $u = 1+\theta$ $du = d\theta$

$$\int \frac{\sqrt{u^2+1}}{u^4}\, du$$

Pg 435
Summary

(39) Integration through x = 0 fctn not defined there.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$\lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^2}\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x^2}\,dx =$$

$$\lim_{b\to 0^-} -\frac{1}{x}\Big|_{-1}^{b} + \lim_{a\to 0^+} -\frac{1}{x}\Big|_{a}^{1} =$$

$$\lim_{b\to 0^-} \left(-\frac{1}{b}+1\right) + \lim_{a\to 0^+} \left(-1+\frac{1}{a}\right) =$$ Both diverge,

(40) $$\int_{0}^{\infty} \frac{x^{-\frac{1}{2}}}{\boxed{1+x}}\,dx = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

let $u = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$ $$2\int_{0}^{\infty} \frac{du}{\boxed{1+u^2}} = 2\lim_{b\to\infty} \int_{0}^{b} \frac{du}{1+u^2} =$$

$x = 0$ $u = 0$

$x = \infty$ $u = \infty$

$$2\lim_{b\to\infty} \tan^{-1} u\Big|_{0}^{b} = 2\lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = 2\cdot\frac{\pi}{2} = \pi$$

(41) $\int_{0}^{1} x^4 \ln x\, dx$ $u = \ln x$ $v = \frac{1}{5}x^5$

$du = \frac{1}{x}\,dx$ $dv = x^4$

$$\lim_{a\to 0} \frac{1}{5}x^5 \ln x\Big|_{a}^{1} - \frac{1}{5}\int_{0}^{1} x^4\,dx = -\frac{1}{25}x^5\Big|_{0}^{1} = \frac{-1}{25}$$

$$\lim_{a\to 0} \frac{1}{5}a^5 \ln a$$

$$\frac{1}{5}\frac{\ln a}{a^5} \quad \frac{\infty}{\infty}$$

$$\lim_{a\to\infty} \frac{1}{5}\frac{\frac{1}{a}}{5a^4} \Rightarrow 0$$

42

a) $G(a) = \int_0^{\infty} \frac{1}{(1+x^a)(1+x^2)}\,dx$

$$G(0) = \int_0^{\infty} \frac{1}{2(1+x^2)}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2} \tan^{-1} x \Big|_0^b$$

$$\lim_{b\to\infty} \frac{1}{2}\left(\tan^{-1} b - \tan^{-1} 0\right)$$

$$\frac{1}{2}\cdot\frac{\pi}{2} =$$

$$\frac{\pi}{4}$$

$G(1) = \int_0^{\infty} \frac{1}{(1+x)(1+x^2)}\,dx$ partial fractions.

$\lim_{b\to\infty} \int_0^b \frac{1}{1+x}\cdot\frac{1}{1+x^2}\,dx \quad \frac{1}{(x)} = \frac{a}{1+x} + \frac{bx+c}{x^2+1}$

$$1 = a(x^2+1) + (bx+c)(x+1)$$

$x=-1 \quad 1 = 2a \quad \therefore a = \frac{1}{2}$

$x=0 \quad 1 = a + c \quad \therefore c = \frac{1}{2}$

$x=1 \quad 1 = 2a + 2(b+c)$

$1 = 1 + 2b + 2c$

$b = -c \quad \therefore b = -\frac{1}{2}.$

$$\lim_{b\to\infty} \int_0^b \frac{1}{2}\frac{1}{1+x}\,dx + \lim_{b\to\infty} \frac{1}{2}\int_0^b \frac{-x+1}{x^2+1}\,dx$$

$$\lim_{b\to\infty} -\frac{1}{2}\frac{1}{2}\int_0^b \frac{2x}{x^2+1}\,dx + \lim_{b\to\infty} \frac{1}{2}\int_0^b \frac{1\,dx}{x^2+1}$$

$$\lim_{b\to\infty}\left(\frac{1}{2}\ln|1+x|\Big|_0^b - \frac{1}{4}\ln|x^2+1|\Big|_0^b + \underbrace{\frac{1}{2}\tan^{-1}b}_{\frac{\pi}{4}}\right.$$

$$\lim_{b\to\infty}\left(\frac{1}{4}\ln\frac{(1+b)^2}{b^2+1} - \frac{1}{4}\ln 1\right.$$

$$\downarrow$$

$$\lim_{b\to\infty}\frac{1+2b+b^2}{b^2+1}$$

$$\lim_{b\to\infty}\frac{\frac{1}{b^2}+\frac{2}{b}+1}{1+\frac{1}{b^2}} \to 1$$

$$\downarrow$$

$$0 - \frac{1}{4}\cdot 0 + \frac{\pi}{4} = \frac{\pi}{4}.$$

$$G(2) = \int_0^{\infty} \frac{1}{(1+x^2)^2}\,dx$$

$$\lim_{b\to\infty} \int_0^b \frac{dx}{(1+x^2)^2}$$

$$1 = \frac{ax+b}{1+x^2} + \frac{cx+d}{(1+x^2)^2}$$

$$1 = (ax+b)(1+x^2) + cx+d$$

$a = 0$
$b = 0$
$c = 0$
$d = 1$

$x = 0$ $\quad 1 = b+d$

$x = 1$ $\quad 1 = 2(a+b)+c+d$

$x = -1$ $\quad 1 = 2(-a+b)-c+d$

$1 = -2a+2b-c+d$

$1 = 2a+2b+c+d$

$2 = 4b+2d$

$1 = 2b+d$

$-1 = -b-d$

$0 = b \therefore b = 0 \therefore d = 1$

$x = 2$ $\quad 1 = (2a+b)5 + 2c+d$

$0 = 10a+2c$

$5a = -c$

oops went in a circle somewhere.

$1 = 2(a+b)+c+d$

$1 = 2a-5a+1$

$0 = -3a$

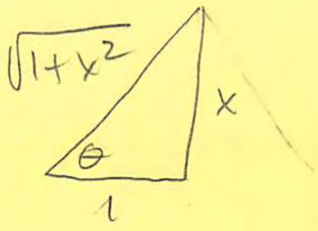
$a = 0$

try again NEXT Page

42

pg 434

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx$$



$$\frac{1}{(\sqrt{1+x^2})^4}$$

let $x = \tan\theta$ $\quad x=0 \;\; \theta=0$

$dx = \sec^2\theta \, d\theta$ $\quad x=\infty \;\; \theta=\frac{\pi}{2}$

$$\lim_{b\to\frac{\pi}{2}} \int_0^b \frac{\sec^2\theta \, d\theta}{\sec^4\theta}$$

$$\lim_{b\to\frac{\pi}{2}} \int_0^b \cos^2\theta \, d\theta$$

$$\lim_{b\to\frac{\pi}{2}} \frac{1}{2}\int_0^b 1+\cos 2\theta \, d\theta$$

$$\lim_{b\to\frac{\pi}{2}} \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^b$$

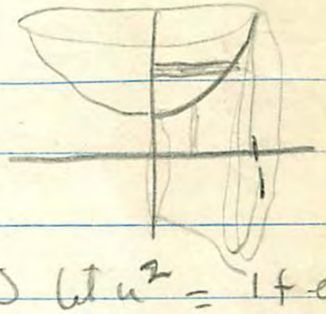
$$\lim_{b\to\frac{\pi}{2}} \frac{1}{2}\left(b + \frac{1}{2}\sin 2b\right)$$

$$\frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{2}\cdot 0\right)$$

$$\frac{\pi}{4}$$

P 4
11, e

(43) $Y = e^x$ $x \in [0,1]$



arc length
(elementary) $\int_0^1 \sqrt{1+e^{2x}}\,dx$ $\rightarrow$ let $u^2 = 1+e^{2x}$

$2u\,du = 2e^{2x}\,dx$

but $e^{2x} = u^2-1$ etc.

Rotate about x axis radius e^x

elementary $2\pi\int_0^1 e^x\sqrt{1+e^{2x}}\,dx$

Rotate about y axis radius $x = \ln y$

not elementary. $2\pi\int_0^1 x\sqrt{1+e^{2x}}\,d\underline{x} = 2\pi\int_1^e x\sqrt{1+(Y')^2}$

chosen because we can describe the length easily

$2\pi\int_0^1 x^2\sqrt{1+e^{2x}}$

$x = \ln Y$ $dx = \frac{1}{Y}dy$ $dY = x\,dx$

b) $2\pi\int_0^1 e^x\sqrt{1+e^{2x}}\,dx$ let $u = e^x$ $du = e^x\,dx$ $x=0$ $u=1$ $x=1$ $u=e$

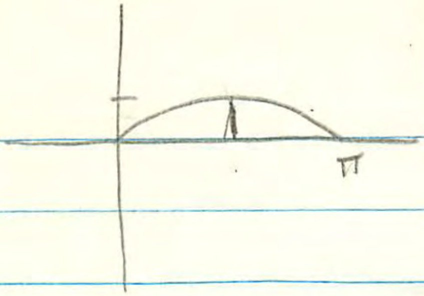
$2\pi\int_1^e \sqrt{1+u^2}\,du$ ends up w/ $\int\sec^3\theta$ ie $\frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln(\sqrt{1+x^2}+x)$ Pg 352

$Y=e^x$
$x \in [0,1]$
$Y \in [1,e]$

$\int_1^e \ln y\sqrt{\quad}\,dy$ $\leftarrow$ what goes in here ie the length of the curve in terms of Y rather that x as usual.

p 435
&
Summary

(44) $y = \sin x$ $x \in [0, \pi]$



$y' = \cos x$

arc length $\int_0^{\pi} \sqrt{1 + \cos^2 x}\ dx$

Non elementary

rotate about x axis. radius $\sin x$

$$2\pi \int_0^{\pi} \sin x \sqrt{1 + \cos^2 x}\ dx$$

rotate about y axis radius is x Non elementary.

$$2\pi \int_0^{\pi} x \sqrt{1 + \cos^2 x}\ dx$$

pg 435

(45) $\int \sqrt{1+a\sin^2\theta}\, d\theta$ if $a=0$ $\int 1\, d\theta = \theta + c$

if $a=-1$ then $\int \cos\theta\, d\theta = -\sin\theta + c$

only. needed to make the trig substitution

(46) $\int \frac{\sqrt{1+ax^2}}{\sqrt{1-x^2}}\, dx$ if $a=0$ $\int \frac{dx}{\sqrt{1-x^2}}$ let $\sin^2\theta = x$ etc.

if $a=-1$ then $\int dx = x$,

(47) $\int \sqrt{1+b\cos\theta}\, d\theta$ if $b=0$ obvious.

if $b=+1$ then the trig sub would be $\cos\left(\frac{\theta}{2}\right)$

if $b=-1$ then we have $\sin\left(\frac{\theta}{2}\right)$.

(48) $\int \frac{\sqrt{1+bx}}{\sqrt{1-x^2}}\, dx$ 0 is obvious. if $b=\pm 1$ then function reduces to

$$\sqrt{\frac{1+x}{1-x^2}} = \sqrt{\frac{1}{1-x}} \quad \text{or} \quad \sqrt{\frac{1-x}{1-x^2}} = \sqrt{\frac{1}{1+x}}$$

(57) $\int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{x^2\, dx}{1+x^4}$ let $x = \frac{1}{y}$ $dx = \frac{-1}{y^2}\, dy$

$x=0$ $y \to \infty$

$x=\infty$ $y \to 0$

$$-\int_0^\infty -\frac{1}{y^2}\, dy \left(\frac{1}{1+\frac{1}{y^4}}\right) = \int_0^\infty \frac{1}{y^2}\, dy \left(\frac{1}{\frac{y^4+1}{y^4}}\right) = \int_0^\infty \frac{y^2\, dy}{y^4+1}$$

disagrees w/ answer book.

Pg 4

(54) $r = 1 + K\cos\theta$ arc length. $\sqrt{r^2 + (r')^2}$

$r' = -K\sin\theta$

$$1 + 2K\cos\theta + \underbrace{K^2\cos^2\theta + K^2\sin^2\theta}_{K^2}$$

$$\sqrt{(1+K^2) + 2K\cos\theta} = \sqrt{(1+K^2)}\sqrt{1 + \frac{2K}{1+K^2}\cos\theta}$$

$$\frac{2K}{1+K^2} \overset{st}{=} \begin{cases} -1, & K=-1 \\ 0, & K=0 \\ 1, & K=1 \end{cases}.$$

two hard. (55) $\frac{dv}{dt} = -\sqrt{Y}$ $V(Y) = \pi r^2 Y$ $\frac{dV}{dt} = \pi r^2 \frac{dY}{dt}$

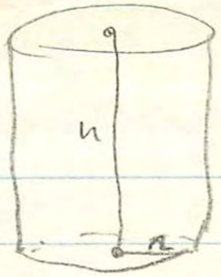
a) find T when $Y = \frac{h}{2}$

$$-\sqrt{Y} = \pi r^2 \frac{dY}{dt} \Rightarrow dT = -\frac{\pi r^2}{\sqrt{Y}} dY$$

in order to determine T from dT we integrate

435

55



$\frac{dV}{dt} = -\sqrt{y}$

$\pi r^2\sqrt{h}(2-\sqrt{2})$

$2\pi r^2\sqrt{h}$

$V(h) = \pi r^2 h$

$V(y) = \pi r^2 y$

$\frac{dV}{dt} = \pi r^2 \frac{dy}{dt} = -\sqrt{y}$

$\frac{1}{\sqrt{y}}\frac{dy}{dt} = \frac{-1}{\pi r^2}$

$\frac{dV}{\sqrt{y}} = -dt$

$2y^{\frac{1}{2}} = \frac{-t}{\pi r^2}$

$-2\sqrt{y} = -t + c$

$t=0$ $-2\sqrt{h} = 0 + c$
$y=h$

$\therefore c = -2\sqrt{h}$

$2\sqrt{y} = t - 2\sqrt{h}$

$V\left(\frac{h}{2}\right) = \frac{\pi r^2 h}{2}$

$dV = -\sqrt{y}\,dt$

$V = -\frac{2}{3}y^{\frac{3}{2}} + c$

(51) $Y = \frac{\ln x}{x^2}$ $\lim_{x\to\infty} Y = 0$ ∴ fctn ~~crosses~~ Touches x axis and the area is finite

Y Never < 0.

(52) $\int_0^1 x \ln x \, dx$ Not defined at $x = 0$.

Consider $\int_0 \overset{u}{x} \overset{v}{\ln x} \, dx$ $u = \ln x$ $v = \frac{1}{2}x^2$

$du = \frac{1}{x}$ $dv = x\,dx$

$\frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$

$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$

$\lim_{a\to 0^+} \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \Big|_a^1 = \lim_{a\to 0^+} \left(0 - \frac{1}{4} - \frac{a^2}{2}\ln a + \frac{a^2}{4}\right)$

$\lim_{a\to 0^+} \left(-\frac{1}{4} + \frac{a^2}{4}\right) = -\frac{1}{4}$ $\lim_{a\to 0^+} \frac{a^2 \ln a}{2} =$

$\lim_{a\to 0^+} \frac{\ln a}{\frac{2}{a^2}} = \frac{1}{2}\lim_{a\to 0^+} \frac{\ln a}{a^{-2}}$ use l'hopital's Rule.

$\frac{1}{2}\lim_{a\to 0^+} \frac{\frac{1}{a}}{-2a^{-3}} = \frac{1}{2}\lim_{a\to 0^+} \frac{-a^2}{2} = 0$ so!

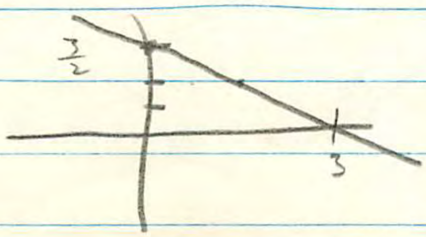
the limit exists ∴ the integral is convergent.

(53) $r = \frac{3}{\cos\theta + 2\sin\theta}$ $r\cos\theta = x$ $r\sin\theta = y$

so divide by r

$1 = \frac{3}{r\cos\theta + 2r\sin\theta} = 1 = \frac{3}{x + 2y} \Rightarrow x + 2y = 3$

$Y = -\frac{1}{2}x + \frac{3}{2}$



EASY

ints by partial fra[illegible]
as extra

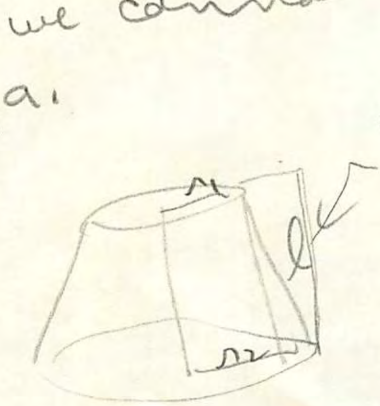
~~Explain how the~~

~~formula $2\pi\int r\,dx$ and~~

~~The formula $2\pi\int_a^b r\,ds$~~

We have ~~seen integrations involving shells to get volume~~
has a ~~similar set up when setting~~ surface area, that
is ~~revolving~~ a line rather than a shell.

Explain why we can add circles to get volume
but we cannot add circumferences to get surface
area.



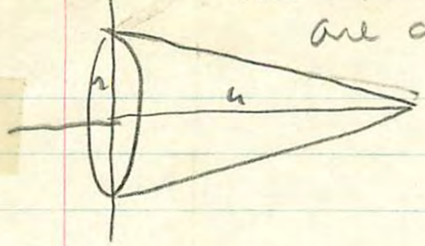
$S = \pi(r_1+r_2)l$ lateral area

This is portion of a
cone called
frustum

$2\pi r$ method
circum of circle
assumes $r_1 = r_2$

If $r_1 = r_2$ then $2\pi r l$ ie cylinder.
surface area.

The following should be compared with the text to see that the same results are achieved in a non calculus method. 415-421 10.11



given a cone with hieght h and radius r find the surface area.

This really could be re stated:
given a line $y = \frac{-r}{h}x + r$. Revolve it around the x axis. The line is bounded in the 1st Quadrant.

What we want to do is sum up each of the lines formed when we rotate our given line. The formula we need is.

$$\int_a^b 2\pi y \sqrt{1+(y')^2}\,dx$$

this revolves our line About THIS AXIS

This is a TYPICAL radius of Revolution

this is the length of the curve we are rotating.

for our pblm the limits are fm 0 to h. a typical radius of revolution is $y = \frac{-rx}{h} + r$, $y' = -\frac{r}{h}$
So we get

$$\int_0^h 2\pi\left(-\frac{rx}{h}+r\right)\sqrt{1+\frac{r^2}{h^2}}\,dx =$$

$$2\pi\int_0^h\left(-\frac{rx}{h}+r\right)\frac{\sqrt{h^2+r^2}}{h}\,dx = \frac{2\pi\sqrt{h^2+r^2}}{h}\int_0^h\left(-\frac{rx}{h}+r\right)dx =$$

$$\frac{2\pi\sqrt{r^2+h^2}}{h^2}\left(-\frac{rx^2}{2}+rhx\right)\Big|_0^h = \frac{2\pi\sqrt{r^2+h^2}}{h^2}\left(\frac{-rh^2}{2}+rh^2\right)$$

$$\frac{2\pi\sqrt{r^2+h^2}}{h^2}\left(\frac{rh^2}{2}\right) = \pi r\sqrt{r^2+h^2}$$

Contrast w/Shell on following Pg